Planning the Unit

Unit 5

Students continue to expand their repertoire of functions in this unit, while at the same time deepening and extending previously learned concepts such as graphing using transformations, algebraic manipulation of expressions and equations, and inverse relations and functions. The unit opens with an exploration of inverses and composition of functions where students discover that not all functions are invertible. Students then explore radical and rational functions including inverse variation models in a variety of contexts and settings. They gain facility with simplifying expressions and solving equations throughout the unit.

Academic Vocabulary

Blackline masters for use in developing students' vocabulary skills are located at the back of this Teacher's Edition. Encourage students to explore the meanings of the academic vocabulary words in this unit, using graphic organizers and class discussions to help students understand the key concepts related to the terms. Encourage students to place their vocabulary organizers in their Math notebooks and to revisit these pages to make notes as their understanding of concepts increases.

Embedded Assessments

The two Embedded Assessments for this unit follow Activities 5.3 and 5.7.

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AP/College Readiness

Unit 5 exposes students to power and rational functions in a variety of contextual settings and expands their knowledge about these types of functions, how to manipulate them, graph them, and apply them to physical settings by:

- Giving students a first opportunity to consider functions for which the domain is a subset of the set of real numbers.
- Studying the asymptotic behavior of rational functions and laying a foundation for understanding limits.
- Providing practice simplifying complicated expressions and solving a variety of equations students are likely to encounter in AP courses.
- Using technology as a tool to explore concepts and relationships and well as solve problems.
- Exploring functions numerically, graphically, verbally, and algebraically.
- Creating and using mathematical models based on written descriptions and collected data.

Embedded Assessment 1 **A Mightier Wind**

- Inverse functions
- Composition of functions
- Transformations of $f(x) = \sqrt{x}$
- Square root equations
- Rational exponents

Embedded Assessment 2 **Planning a Prom**

- Analyzing and graphing rational functions
- Solving rational equations
- Rational models and applications

Suggested Pacing

The following table provides suggestions for pacing either a 45-minute period or a block schedule class of 90 minutes. Space is left for you to write your own pacing guidelines based on your experiences in using the materials.

	45-Minute Period	90-Minute Period	Comments on Pacing
Unit Overview	$\frac{1}{2}$	$\frac{1}{4}$	
Activity 5.1	3	$1\frac{1}{2}$	
Activity 5.2	3	$1\frac{1}{2}$	
Activity 5.3	4	2	
Embedded Assessment 1	1	$\frac{1}{2}$	
Activity 5.4	4	2	
Activity 5.5	3	$1\frac{1}{2}$	
Activity 5.6	4	2	
Activity 5.7	3	$1\frac{1}{2}$	
Embedded Assessment 2	1	$\frac{1}{2}$	
Total	$\mathbf{26\frac{1}{2}}$	$\mathbf{13\frac{1}{4}}$	

Unit Practice

Practice Problems for each activity in the unit appear at the end of the unit.

Math Standards Review

To help accustom students to the formats and types of questions they may encounter on high stakes tests, additional problems are provided at the end of the unit. These problems are constructed for multiple choice, short response, extended response, and gridded responses.

Radical and Rational Functions

Unit 5

Unit Overview

In this unit, you will extend your study of functions to radical, rational, and inverse functions and the composition of functions. You will solve rational equations and inequalities as well as equations with rational exponents.

Unit 5 Academic Vocabulary

Add these words and others that you encounter in this unit to your vocabulary notebook.

- complex fraction
- horizontal asymptote
- inverse variation
- one-to-one function
- power function
- rational exponent
- rational function
- vertical asymptote

Essential Questions

- Why is it important to consider the domain and range of a function?
- How are inverse functions useful in everyday life?

EMBEDDED ASSESSMENTS

This unit has two embedded assessments, following Activities 5.3 and 5.7. These assessments will allow you to demonstrate your understanding of inverse functions, the composition of functions, and solving and graphing radical and rational equations.

UNIT 5 OVERVIEW

Unit Overview

Ask students to read the unit overview and review the terms *function*, *rational*, *inverse*, *domain*, and *range*.

Essential Questions

Read the essential questions with students. Remind them to review these questions periodically as they complete the activities in the unit.

Materials

- Graphing calculator

Academic Vocabulary

As students develop fluency with new terms, encourage them to use precise mathematical language in discussions and writing. Continue to monitor their vocabulary logs for completeness.

Embedded Assessments

There are two embedded assessments in this unit with evaluation rubrics. You may want to review skills needed for the assessment with students prior to the beginning of their work.

UNIT 5 GETTING READY

You may wish to assign some or all of these exercises to gauge students' readiness for Unit 5 topics.

Prerequisite Skills

- Operations with fractions (Items 1, 2)
- Factoring (Items 4, 5)
  - Trinomials
  - Difference of squares
- Operations with radicals (Item 6)
- Simplifying polynomials (Item 3)
- Composition of functions (Item 7)
- Inverse functions (Item 8)

Answer Key

1a. $x \neq 0$; division by zero is undefined.

b. $x \neq 1$; if $x = 1$, the denominator of the fraction would be 0, which is undefined.

2a. $\frac{x}{10}$

b. $\frac{6x - 2}{x + 3}$

c. $\frac{7x + 2}{x(x + 1)}$

3a. $6x^3 y^4$

b. $16a^2b^6$

4. $(9x + 5)(9x - 5)$

5. $(2x + 1)(x - 3)$

6. $8x\sqrt{2x}$

7. $10x - 4$

8. C

UNIT 5

Getting Ready

Write your answers on notebook paper or grid paper. Show your work.

1. What values are not possible for the variable x in each expression below? Explain your reasoning.

a. $\frac{2}{x}$ **b.** $\frac{2}{x - 1}$

2. Perform the indicated operation.

a. $\frac{2x}{5} - \frac{3x}{10}$

b. $\frac{2x + 1}{x + 3} + \frac{4x - 3}{x + 3}$

c. $\frac{2}{x} + \frac{5}{x + 1}$

3. Simplify each monomial.

a. $(2x^2y)(3xy^3)$

b. $(4ab^3)^2$

Factor each expression in Items 4–5.

4. $81x^2 - 25$

5. $2x^2 - 5x - 3$

6. Simplify $\sqrt{128x^3}$.

7. Find the composition $f(g(x))$ if $f(x) = 5x - 4$ and $g(x) = 2x$.

8. Which of the following is the inverse of $h(x) = 3x - 7$?

a. $7 - 3x$

b. $3x + 7$

c. $\frac{x + 7}{3}$

d. $\frac{1}{3x - 7}$

Composition of Functions
Code Breakers

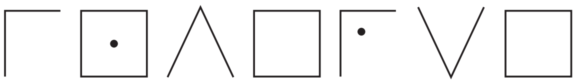
ACTIVITY 5.1

SUGGESTED LEARNING STRATEGIES: Shared Reading, Marking the Text, Summarize/Paraphrase/Retell, Look for a Pattern, Quickwrite

The use of *cryptography* goes back to ancient times. In ancient Greece, Spartan generals exchanged messages by wrapping them around a rod called a scytale and writing a message on the adjoining edges. The Roman general and statesman Julius Caesar used a transposition cipher that translated letters three places forward in the alphabet. For example, the word CAT was encoded as FDW.

In modern times, cryptography was used to secure electronic communications. Soon after Samuel F.B. Morse invented the telegraph in 1844, its users began to *encode* the messages with a secret code, so that only the intended recipient could *decode* them. During World War II, British and Polish cryptanalysts used computers to break the German Enigma code so that secret messages could be deciphered.

Many young children practice a form of cryptography when writing notes in secret codes. The message below is written in a secret code.



1. Try to decipher the seven-letter word coded above.
The word is INVERSE.

2. What do you need to decipher the seven-letter word?
A key or cipher is needed to decode the message.

My Notes

CONNECT TO SCIENCE

Cryptography is the science of code-making (encoding) and *cryptanalysis* is the science of code-breaking (decoding).

CONNECT TO COMPUTING

Modern computers have completely changed the science of cryptography. Before computers, cryptography was limited to two basic types: transposition, rearranging the letters in a message, and substitution, replacing one letter with another. The most sophisticated pre-computer codes used five or six operations. Computers can now use thousands of complex algebraic operations to encrypt messages.

CONNECT TO Cryptography

Student projects (such as creating a scytale) or student reports could be developed from the introduction. The first documented examples of cryptography date back to 1900 BCE. Julius Caesar, Sir Francis Bacon, and Thomas Jefferson are well-known users of cryptography. A form of Julius Caesar's code is used in Item 4. The Braille alphabet and the use of quilts as a means of protecting the Underground Railroad during the Civil War could also be investigated in student reports. Students may be familiar with best-selling novels and popular movies that use cryptography as a central theme. Sarah Flannery's book *In Code: A Mathematical Adventure* might also be of interest to students.

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ACTIVITY 5.1 Investigative

Composition of Functions

Activity Focus
- Inverse functions and relations
- Composition of functions
- One-to-one functions
- Restricted domain and range

Materials
- Graphing calculator

Chunking the Activity

#1–2	#12	#18–20
#3–4	#13	#21
#5–6	#14–16	#22–23
#7–11	#17	#24–26

TEACHER TO TEACHER You can add visual interest by presenting pictures of a scytale, Julius Caesar, or the Braille alphabet as students read the introduction.

First Four Paragraphs Shared Reading, Marking the Text, Summarize/Paraphrase/Retell In the context of cryptography, students investigate inverse relations and functions, and one-to-one functions numerically, algebraically, and graphically. Linear functions are reviewed and then the properties are extended to all functions.

1 Look for a Pattern The cipher used to encode the message, called the Pigpen Cipher, is shown at the bottom of the next page. Allow students time to try to decipher the message before giving them the code.

2 Quickwrite Based on the introduction, students should ask for a key or a code relating the letters and symbols.

3 Look for a Pattern, Quickwrite Given a little time, most students will decipher MATH.

4 Look for a Pattern, Create Representations, Debriefing Given that the first two words were academic words, some students are likely to discover the shift of 2 letters and decipher SCHOOL.

Paragraphs and Tables
Marking the Text, Summarize/Paraphrase/Retell

TEACHER TO TEACHER Students should have the opportunity to decipher the messages without prompting. Some students will be able to decipher each message. They will find it much more rewarding if they can discover the code on their own. To make sure that students don't look ahead, you may want to distribute a copy of only Items 1–4. Do not spend too much time on these early Items. They are designed only to present a setting in which to study the mathematics.

The code A = 1, B = 2, and so on is used throughout the rest of this activity to convert letters to numbers.

My Notes

SUGGESTED LEARNING STRATEGIES: Look for a Pattern, Quickwrite, Create Representations, Marking the Text, Summarize/Paraphrase/Retell

3. The following message uses a numerical code. Can you decode the four-letter word? Explain how you know.

13 1 20 8

The word is MATH. It uses the letter-to-number code.

4. What is this six-letter word?

21 5 10 17 17 14

The word is SCHOOL.

LETTER-TO-NUMBER CODES

A = 1	N = 14
B = 2	O = 15
C = 3	P = 16
D = 4	Q = 17
E = 5	R = 18
F = 6	S = 19
G = 7	T = 20
H = 8	U = 21
I = 9	V = 22
J = 10	W = 23
K = 11	X = 24
L = 12	Y = 25
M = 13	Z = 26

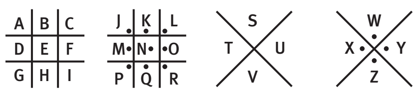
In Item 3, a single function was used to encode a word. The function assigned each letter to the number representing its position in the alphabet.

In Item 4, two functions were used to encode a word. The first function assigned each letter to the number representing its position in the alphabet x and then the function $f(x) = x + 2$ was used to *encode* the message further as shown in the table.

LETTER	x	$f(x)$
S	19	21
C	3	5
H	8	10
O	15	17
O	15	17
L	12	14

CONNECT TO the Pigpen Cipher

This code may be unfamiliar to most students. The cipher shown below illustrates the possibility of encoding a message using symbols rather than numbers or other letters.



Using this code, the letter J would be

SUGGESTED LEARNING STRATEGIES: Create Representations, Look for a Pattern, Work Backward

My Notes

5. Write a function g that could *decode* the message in Item 4 and use it to complete the table below.

$g(x) =$ $x - 2$

x	$g(x)$	LETTER
21	19	S
5	3	C
10	8	H
17	15	O
17	15	O
14	12	L

6. Try to decipher the more difficult message below. First, each letter in the message was assigned a number based on its position in the alphabet, and then another function encoded the message further.

20 −1 50 8 11 50

The word is HARDER.

7. The encoding function for Item 6 is $f(x) =$ $3x - 4$.

Write a decoding function g and complete the table below.

$g(x) = \frac{x+4}{3}$

x	$g(x)$	LETTER
20	8	H
−1	1	A
50	18	R
8	4	D
11	5	E
50	18	R

ACTIVITY 5.1 *Continued*

5 Create Representations The encoding function has already been given to students in the previous Item. They may think that it seems unnecessary to write a decoding function, particularly for such an easy encoding function. Students will understand the efficiency of a decoding function as the encoding functions become more complex and the messages become longer.

TEACHER TO TEACHER Students having difficulty may find it easier to write a verbal representation of the function first and then generalize it to a rule. For example, to undo adding 2, one would need to subtract 2. After describing the process in words, have students try to write the rule.

6 Look for a Pattern, Create Representations Do not expect students to be able to decode this function without using the encoding function. Focus students on determining the decoding function rather than on trying to guess the encoding function. Students should recognize that even a very simple linear function can make guessing the encoding function almost impossible.

7 Create Representations, Work Backward Have students first record the encoding function used in Item 6 in their books. Extending the notion of a verbal rule that would undo $f(x) = 3x - 4$, students might incorrectly suggest that to undo the function and obtain the inverse, one would divide by 3 and then add 4. With two operations, it is important to undo the operations in the reverse order (add 4 and then divide by 3). The correct inverse is $g(x) = \frac{x+4}{3}$, rather than $g(x) = \frac{x}{3} + 4$.

Paragraph Interactive Word Wall, Vocabulary Organizer

9 Create Representations, Simplify the Problem It is not enough to show that the definition of *inverse function* holds true for a single value of x (that is, to show that $f(g(3)) = 3$ and $g(f(3)) = 3$). It is also not sufficient to show that only one of the two parts of the definition is true for all x. Try a numerical approach for students who are having difficulty. For example,

$$1 \xrightarrow{f} -1 \xrightarrow{g} 1$$

$$1 \xrightarrow{g} \frac{5}{3} \xrightarrow{f} 1$$

10-11 Think/Pair/Share, Debriefing These Items emphasize the use of inverse notation.

Suggested Assignment

CHECK YOUR UNDERSTANDING
p. 276, #1a

UNIT 5 PRACTICE
p. 325, #1–2

SUGGESTED LEARNING STRATEGIES: Guess and Check, Interactive Word Wall, Vocabulary Organizer, Create Representations, Simplify the Problem, Think/Pair/Share

My Notes

MATH TIP

$f(g(x))$ indicates a **composition of functions,** in which the range of function g becomes the domain for function f.

CONNECT TO AP

The notion of inverse is a very important concept in the study of functions in advanced mathematics courses.

READING MATH

The symbol for the inverse of function f is f^{-1}. You read this symbol as "f inverse."

8. Verify that your function works by encoding the letter $C = 3$ with f and then decoding it by using g.

$f(3) = 5$

$g(5) = 3$

Recall that functions f and g are called **inverse functions** if and only if $f(g(x)) = x$ for all x in the domain of g and $g(f(x)) = x$ for all x in the domain of f.

9. Use the definition of inverse functions to show that the encoding function $f(x) = 3x - 4$ and its decoding function g are inverses.

If the functions are inverses then their composition will equal x.

$f(g(x)) = 3\left(\frac{x+4}{3}\right) - 4 = x$

$g(f(x)) = \frac{3x - 4 + 4}{3} = x$

10. What is f^{-1} for the function $f(x) = x + 2$?

$f^{-1}(x) = x - 2$

11. What is f^{-1} for the function $f(x) = 3x - 4$?

$f^{-1}(x) = \frac{x+4}{3}$

CONNECT TO AP

One way to think of functions and their inverses is to consider that a function and its inverse "undo" each other. This leads students to the idea that the composition of a function and its inverse leads to the identity function. For example, $f(x) = 2x - 3$ multiplies x by 2 and subtracts 3. The inverse function $g(x)$ would add 3 to x and divide the result by 2 $\left(\text{i.e., } g(x) = \frac{x+3}{2}\right)$. This leads to $f(g(x)) = g(f(x)) = x$. Similarly, inverse relationships will be considered when studying exponential and logarithmic functions and trigonometric functions. In calculus, students will discover a similar relationship between a particular function's derivative and a related anti-derivative.

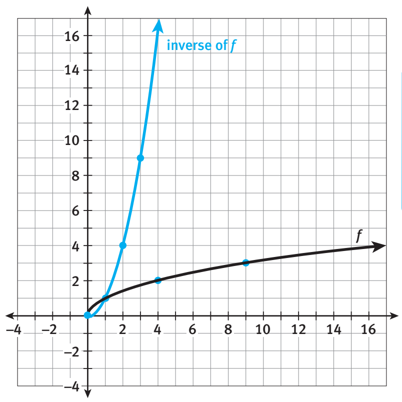
SUGGESTED LEARNING STRATEGIES: Create Representations, Quickwrite, Look for a Pattern, Group Presentation

My Notes

So far, the functions in this activity have been linear functions. Other types of functions also have inverses.

12. The graph of $f(x) = \sqrt{x}$ is shown.

a. List four points on the graph of f and four points on its inverse.
Points on f: (0, 0), (1, 1), (4, 2), (3, 9), points on the inverse (0, 0), (1, 1), (2, 4), (9, 3).



b. Use the points from Part (a) to graph the inverse of f.

MATH TIP

Recall that if (a, b) was a point on the graph of a function then (b, a) must be a point on the graph of the inverse of the function.

c. Find f^{-1} algebraically.
Students will likely get the partial answer shown below. The complete answer has the domain of the inverse restricted to $x \geq 0$.
$f(x) = \sqrt{x}$
$y = \sqrt{x}$
$x = \sqrt{y}$
$x^2 = y$
$f^{-1}(x) = x^2$

d. Graph f and f^{-1} on a calculator. Is f^{-1} on your calculator the same as the graph in Part (b)? Explain below.
Answers will vary. Sample response: the graph of $y = x^2$ has a domain of all real numbers but the inverse we graphed in part *b* is only defined for $x \geq 0$.

e. What are the domain and range of f?
domain: $x \geq 0$. range: $y \geq 0$.

f. Based on your answer in Part (b), what should be the domain and range of f^{-1}?
domain: $x \geq 0$. range: $y \geq 0$.

g. Use your results from Part (f) to complete the following.

$f^{-1}(x) = \underline{\quad x^2 \quad}$ for $x \geq \underline{\ 0\ }$

CONNECT TO Inverses

You may want to review the definition of and properties of inverse functions that students learned in Activity 1.5. If needed, you can also review the process to find the inverse of a relation algebraically.

$f(x) = x + 2$

$y = x + 2$	Let y represent $f(x)$.
$x = y + 2$	Interchange x and y to form the inverse relationship.
$y = x - 2$	Solve for y to find the inverse. (Assume the inverse is a function.)
$f^{-1}(x) = x - 2$	This is the inverse function.

ACTIVITY 5.1 *Continued*

TEACHER TO TEACHER Functions other than linear functions appear in this section. The square root function, quadratic functions, and cubic functions are used to reaffirm the properties of inverse functions. These functions are also used to introduce one-to-one functions and inverse functions with restricted domains.

12 b Create Representations Make sure that student graphs show only the first quadrant and include (0, 0).

12 c Many students might not recognize that the domain is restricted when finding the inverse algebraically. Their thinking will be corrected as they continue through these Items.

12 d Quickwrite Students should recognize that the domain of the inverse needs to be limited. If the function $y = x^2$ is graphed without restricting the domain, the resulting graph will not match the graph made in Item 12b.

12 e–f Look for a Pattern Based on their previous knowledge, students should know that the range of a function is the same as the domain of the inverse, and the domain of a function is the same as the range of its inverse.

12 g Group Presentation, Debriefing Make sure students understand the need to restrict the domain of the inverse before continuing the activity.

TECHNOLOGY TIP

Students may graph functions with a restricted domain on their calculator. Using Boolean algebra, students can enter $y = \frac{x^2}{(x \geq 0)}$.

ACTIVITY 5.1 *Continued*

First Paragraph Interactive Word Wall, Vocabulary Organizer

13 a Create Representations Students should find a few points on g and then use those to graph the inverse.

13 b Quickwrite The inverse of *g* is not a function. Acceptable student answers may include the following:

- A reference to the vertical line test
- For any $x > 4$, there are two corresponding *y*-values
- An example, such as: if $x = 5$, then $y = 1$ or $y = -1$

13 c Look for a Pattern At this point, it is expected that students will understand that they do not need to see the graph of the inverse in order to determine whether it is a function. Students should have discovered a *horizontal line test*. If every horizontal line intersects the graph of the function in at most one point, then the function is one-to-one (and its inverse will be a function). These definitions are presented on the next page of the student edition.

13 d Work Backward, Debriefing $x = y^2 + 4$ is the inverse relation of *g*. This statement can also be written as $y = \pm\sqrt{x - 4}$.

SUGGESTED LEARNING STRATEGIES: Interactive Word Wall, Vocabulary Organizer, Create Representations, Quickwrite, Look for a Pattern, Work Backward

My Notes

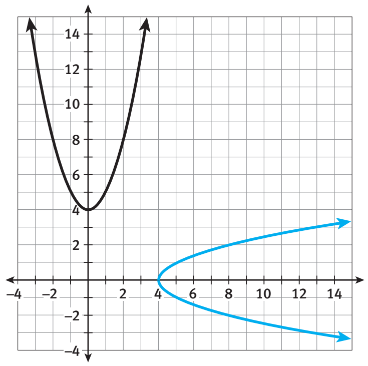
MATH TERMS

A **relation** is a set of ordered pairs that may or may not be defined by a rule. Not all relations are functions, but all functions are relations.

All functions have an inverse *relation*, but the relation may or may not be a function.

13. Use the quadratic function *g* graphed below.

a. Graph the inverse of *g*.



b. Is the inverse of *g* a function? Explain your reasoning.
Explanations will vary. Sample response: The inverse of *g* is not a function because it has 2 *y*-values for at least one *x*-value.

c. What characteristic of the graph of a function can you use to determine whether its inverse relation is a function?
Answers will vary. Sample response: If a function has 2 *x*-values for at least one *y*-value then its inverse relation will not be a function.

d. The quadratic function shown in the graph is $g(x) = x^2 + 4$. Find an equation for the inverse relation of this function.
The equation is $x = y^2 + 4$, or $y^2 = x - 4$; $y = \pm\sqrt{x - 4}$ for $x \geq 4$

SUGGESTED LEARNING STRATEGIES: Interactive Word Wall, Vocabulary Organizer, Quickwrite, Create Representations

A function is defined as **one-to-one** if, for each number in the range of the function, there is exactly one corresponding number in the domain of the function.

14. Is g from Item 13 a one-to-one function? Explain.
According to the definition, the function g is not one-to-one because each y (except $y = 4$) has 2 different x-values associated with it.

15. What do you know about a function whose inverse relation is a function?
That function will have to be one-to-one.

16. Determine whether each type of function will *always* have an inverse that is a function. Explain your reasoning.

a. Linear function
Linear functions ($m \neq 0$) are one-to-one, so they will always have an inverse that is a function.

b. Quadratic function
Quadratic functions are not one-to-one, so they will never have an inverse that is a function.

17. Investigate the cubic function $h(x) = x^3 - 6x^2 + 8x + 5$.

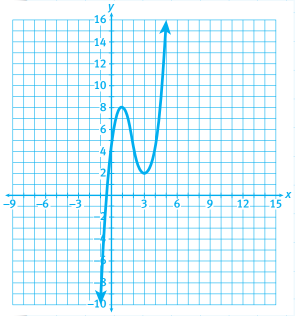
a. Use your calculator to graph h in the viewing window $[-10, 16]$ by $[-10, 16]$ and sketch the results in the My Notes section.

b. Explain whether or not h is a one-to-one function.
The function is not one-to-one. For example, there are 3 points that contain $y = 6$.

My Notes

ACADEMIC VOCABULARY

A one-to-one function is a function in which for each element of the range, there is exactly one corresponding element in the domain.



ACTIVITY 5.1 *Continued*

First Paragraph Interactive Word Wall, Vocabulary Organizer

14 Quickwrite

15 Quickwrite

16 Quickwrite, Debriefing
The inverses of all linear functions (where $m \neq 0$) are functions, because they are one-to-one. Students are likely to overlook the special case of the linear function where y is equal to a constant and where the inverse is *not* a function. No inverses of any quadratic functions will be functions, because quadratic functions are never one-to-one.

17 **a–b** Create Representations, Quickwrite Check students' graphs to be sure they sketched correctly what they found on the graphing calculator.

17 c–d Look for a Pattern, Debriefing Students may quickly answer "Yes" to Part (c) without investigating additional cubic functions. Part (d) may help these students to catch their error.

Suggested Assignment

CHECK YOUR UNDERSTANDING
p. 276, #1b, 2–3

UNIT 5 PRACTICE
p. 325, #3–4

TEACHER TO TEACHER On this day, students return to the context of cryptography. Students should be familiar with the mechanics of working with composite functions—if not, review content from Activity 1.4. The context will help students improve their intuitive understanding of composite functions. They will also encode messages with functions that are not one-to-one to observe the difficulties of decoding these messages. Students should enjoy the challenges of this section, but it can be omitted if necessary due to time constraints.

18 Create Representations

19 Create Representations

20 Create Representations, Quickwrite Instead of first encoding the numbers and then encoding them a second time, composition allows the two processes to be done in a single step. Students will realize how quickly the composite function h does the work of the other two functions by using it to encode the original message.

SUGGESTED LEARNING STRATEGIES: Look for a Pattern, Create Representations, Quickwrite

My Notes

17. *(continued)*

c. Is your answer in Part (b) true for all cubic functions? Explain.
No. $y = x^3$ is a one-to-one cubic function.

d. Does a cubic function always have an inverse that is a function? Explain your reasoning.
No. Some cubic functions are one-to-one and some are not.

Three students are encoding and decoding messages that begin with the numerical code used in Item 3. Suppose that the message HELLO is to be passed from one student to a second student and then on to a third student. Use this information for Items 18–24.

18. The first student translates HELLO to numbers and then encodes it with the function $f(x) = -2x + 12$. What encoded message will the second student receive?

H	E	L	L	O
8	5	12	12	15
–4	2	–12	–12	–18

19. After receiving the encoded message, the second student encodes it again, using the function $g(x) = -x + 9$. What encoded message will the third student receive?

–4	2	–12	–12	–18
13	7	21	21	27

20. Let h be the composite function $h(x) = g(f(x))$.

a. Write a rule for h.

$h(x) = g(f(x)) = g(-2x + 12) = -(-2x + 12) + 9$
$= 2x - 12 + 9$
$= 2x - 3$

b. Explain how h relates to the encoding of the message HELLO.
Answers will vary. Sample response: h provides a quicker method for encoding the two-step message. Instead of substituting a value into f and then taking that result and substituting it into g, you only need to substitute and evaluate once when encoding the message with h.

SUGGESTED LEARNING STRATEGIES: Work Backward, Create Representations, Quickwrite, Summarize/Paraphrase/Retell, Look for a Pattern

My Notes

21. The third student receives another message shown below. What does this message say?

27 25 7   37 27   27 25 7

The message says ONE TO ONE. $h^{-1}(x) = \frac{x+3}{2}$

27	25	7		37	27		27	25	7
15	14	5		20	15		15	14	5
O	N	E		T	O		O	N	E

22. Suppose that the message above had been encoded using the composite function $k(x) = f(g(x))$. How would the message have been encoded?

Answers will vary. Sample response: The numbers were substituted into g and the output was then substituted into f.
$K(x) = -2(-x + 9) + 12 = 2x - 6$;
code: 24 22 4   34 24   24 22 4

23. Can the rule found in Item 20(a) also be used for k? Explain.

Answer will vary. Sample response: No. Composition of functions is not commutative. That is in general, $g(f(x)) \neq f(g(x))$.

24. Suppose that the third student is given f and g, and the encoded message. What additional information would help this student to decode the message more efficiently? Explain.

Answers will vary. Sample response: They would need to know the order in which the two functions were composed.

25. Another message is encoded twice. First, the message is encoded with the function $f(x) = 2x - 29$, and then it is encoded with the function $g(x) = x^2$.

a. Encode the word GRAPH using the composite function.

GRAPH is encoded as $f(g(x)) = (2x - 29)^2$ and is shown below.

G	R	A	P	H
7	18	1	16	8
225	49	729	9	169

b. Decode the following message.

1 1 121   1 1 361   121 1   1 1 361

The inverse of the composite function is $y = \frac{29 \pm \sqrt{x}}{2}$ and is shown below.

1	1	121		1	1	361		121	1		1	1	361
14 or 15	14 or 15	9 or 20		14 or 15	14 or 15	5 or 24		9 or 20	14 or 15		14 or 15	14 or 15	5 or 24
N or O	N or O	I or T		N or O	N or O	E or X		I or T	N or O		N or O	N or O	E or X

ACTIVITY 5.1 *Continued*

21 Work Backward, Create Representations Some students may decode the message as a two-step process. They should be encouraged to use the inverse of h so that the message can be decoded in a single step.

22 Quickwrite The second student passed the message to the first student, who then passed the message on to the third student.

23 Quickwrite Students should realize that k and h are not the same functions, and that, in general, the operation of composition of functions is not commutative.

24 Summarize/Paraphrase/Retell, Quickwrite The message could be decoded more efficiently if the third student knew which function was first used to encode the message. Without this information, a student would have to find the inverses of both $f \circ g$ and $g \circ f$. After decoding the message both ways, the student would have to choose the message that makes more sense.

25 Work Backward, Look for a Pattern, Debriefing Because the composite encoding function was not one-to-one, each decoding input has two possible outputs. The message that was actually encoded was NOT ONE TO ONE.

26 Think/Pair/Share, Quickwrite, Debriefing Because h is a function, it can be used to encode messages, as in Item 25. For every letter in a given message, there will be exactly one number in the coded message that is associated with each letter. Since h is not one-to-one, its inverse is not a function. Therefore, the inverse cannot be used to decode messages because some of the numbers in a given encoded message will have two letters associated with the numbers.

Suggested Assignment

CHECK YOUR UNDERSTANDING
p. 276, #4–5

UNIT 5 PRACTICE
p. 325, #5

CHECK YOUR UNDERSTANDING

1a. $f^{-1}(x) = \sqrt[3]{x+6}$, Domain and range of f and f^{-1} are both all real numbers.

b. $f^{-1}(x) = \left(\frac{x}{2}\right)^2 + 5$, for $x \geq 0$.
Domain of f: $x \geq 5$.
Range of f: $y \geq 0$.
Domain of f^{-1} : $x \geq 0$.
Range of f^{-1} : $y \geq 5$

2. The function $f(x) = 10^x$ has an inverse function because it is a one-to-one function. The function $f(x) = x^2 - 10$ has no inverse function because it is not a one-to-one function. The table does not represent a function that has an inverse because there are two x-values (-3 and 0) for a single y-value (-1).

3a. $g(f(x)) = (2x - 3)^2 - 2(2x - 3) - 8$
$= 4x^2 - 16x + 7$

b. The inverse is not a function because quadratic functions are not one-to-one.

SUGGESTED LEARNING STRATEGIES: Think/Pair/Share, Quickwrite

My Notes

26. Consider $h(x) = g(f(x))$.

a. Is h a function?
Yes

b. Is h a one-to-one function?
No

c. How do your answers in Parts (a) and (b) relate to your work in Item 25? Explain.
Because h is a function, it can be used to encode messages, as in Question 25(a). For every letter in a given message, there will be exactly one number in the coded message that is associated with each letter. Since h is not one-to-one, its inverse is not a function. Therefore, decoding messages as in Question 25(b), can be problematic because decoded messages will have two letters associated with each number in the sender's message.

CHECK YOUR UNDERSTANDING

Write your answers on notebook paper. Show your work.

1. Find the inverse of each function. State the domain and range of the function and its inverse.

a. $f(x) = x^3 - 6$

b. $f(x) = 2\sqrt{x-5}$

2. Which functions have inverse functions? Explain your reasoning.

a. $f(x) = 10^x$ **b.** $f(x) = x^2 - 10$

c.

x	-3	-1	0	2
$f(x)$	-1	0	-1	3

3. Given $f(x) = 2x - 3$ and $f(x) = x^2 - 2x - 8$.

a. Find $y = g(f(x))$.

b. Is the inverse of y a function? Explain.

4. Classify each statement as *always*, *sometimes*, or *never* true. Explain your reasoning.

a. The inverse of a linear function is also a function.

b. The inverse of a quadratic function is also a function.

c. The inverse of a cubic function is also a function.

5. MATHEMATICAL REFLECTION How does the graph of a quadratic function demonstrate whether its inverse will be a function or not?

4a. Sometimes, when $m \neq 0$.

b. Never, because quadratic functions are not one-to-one.

c. Sometimes, when the cubic function is one-to-one.

5. Answers may vary. Sample answer: The graph of a quadratic function shows that there are two domain values for a single range value, so the function is not one-to-one.

Graphing Square Root Functions
Go, Boat, Go!

ACTIVITY
5.2

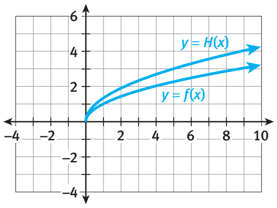
SUGGESTED LEARNING STRATEGIES: Activating Prior Knowledge, Create Representations, Quickwrite

Suppose the hull speed in knots H of a sailboat is given by the function $H(x) = 1.34\sqrt{x}$, where x is the length of the boat in feet at the waterline.

1. The hull speed function is a transformation of the parent square root function $f(x) = \sqrt{x}$.

a. Graph H and f on the same axes. How do these graphs compare to each other?

H is a vertical stretch of f. Both graphs contain the point (0, 0).



b. What are the domain and the range of f?

The domain of f is $x \geq 0$. The range of f is $y \geq 0$.

c. What are the domain and range of H?

The domain and range of H are the same as f.

2. Explain how you could use transformations of the graph of $f(x) = \sqrt{x}$ to graph $g(x) = 2\sqrt{x}$.

Multiply the y-coordinate of the points on the graph of $f(x) = \sqrt{x}$ by 2. This will vertically stretch the parent graph by a factor of 2.

My Notes

CONNECT TO SAILING

The speed of a boat is measured in knots (nautical miles per hour). The distance it travels in water is measured in nautical miles. A nautical mile is equal to 1.15 statute miles.

MATH TIP

To graph the parent square root function, use key points with x-values that are perfect squares, such as 0, 1, 4, and 9.

CONNECT TO Navigation

Historically, a nautical mile is the length of an arc equal to 1 minute $\left(\frac{1}{60^{\text{th}}}\text{ of a degree}\right)$ along a meridian of Earth. The standard used today is 1 nautical mile (NM) = 1852 m (approximately 1.15 miles). The term knot (1 knot = 1 nautical mile per hour) derives from the 'log line' used to measure the speed of a ship before electronic devices were available. A log (wedge-shaped piece of wood) tied to a length of knotted rope was thrown overboard. Knots were tied at even spaces along the line at 47 ft 3 in. intervals. Sailors counted the number of knots that passed over the edge in a 28-second interval.

ACTIVITY 5.2 Guided

Graphing Square Root Functions

Activity Focus

- Transformations of $f(x) = \sqrt{x}$
- Square root equations
- Extraneous solutions

Materials

- Graphing or scientific calculator

Chunking the Activity

#1–2	#8	Example 2
#3–5	#9–11	Try These A
#6–7	Example 1	#12–13

TEACHER TO TEACHER This activity uses a familiar formula from sailing to introduce the parent square root function and related equations. The hull speed equation estimates the theoretical maximum hull speed of single hull displacement type sailboats.

1 Activating Prior Knowledge, Create Representations, Quickwrite Emphasize plotting key points for the parent square root function, such as (0, 0), (1, 1), (4, 2), and (9, 3) when sketching graphs.

Students could use a calculator to make their graphs if needed.

Transformations should already be familiar to students, although this will be the first parent function whose domain is not all real numbers. Make sure to reinforce using proper vocabulary when students write about how the two graphs compare.

2 Quickwrite, Debriefing The vertical stretch factor of 2 implies that each y-value of the parent function will be multiplied by 2 to produce the y-values for g. At this point, students should be proficient at articulating the effect of a vertical stretch.

TEACHER TO TEACHER The definitions for the transformations on this page should be familiar to students and are merely provided for reference. You should not need to re-teach these terms or their effect on graphs of functions. This activity merely extends the use of transformation graphing to a different type of parent function.

3-5 Give students time to work these Items in their groups.

3 Quickwrite

4 Create Representations
Check to see that student graphs are accurate by using the key points. Students should not be making a table of values to graph g. Instead they should simply plot the key points on the graph of f translated 3 units to the right and then sketch in the curve.

5 Think/Pair/Share, Debriefing

6 Think/Pair/Share, Look for a Pattern, Quickwrite Check to see that students are using appropriate vocabulary. Be sure that students recognize how to identify the direction of horizontal translations. The function $f(x + 2) = f(x - (-2))$ translates the graph of f horizontally two unit to the left, not to the right.

See *Connecting Transformation Definitions and Function Notation* on page 279.

Graphing Square Root Functions
Go, Boat, Go!

SUGGESTED LEARNING STRATEGIES: Quickwrite, Create Representations, Think/Pair/Share, Look for a Pattern

My Notes

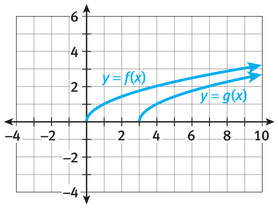
MATH TIP
Recall that the function $y = c \cdot f(x)$ represents a *vertical stretch or shrink* of the original function $y = f(x)$ after the y-values have been multiplied by c.

MATH TIP
Recall that the function $y = -f(x)$ represents a *reflection over the x-axis* of the original function $y = f(x)$ after the y-values have been multiplied by -1.

MATH TIP
Recall that the function $y = f(x + c)$ results in a *horizontal translation* of the original function $y = f(x)$ and $y = f(x) + c$ results in a *vertical translation* of the original function.

3. How does the graph of $g(x) = \sqrt{x - 3}$ compare to the graph of $f(x) = \sqrt{x}$?
It is a horizontal translation of the parent function 3 units to the right.

4. Sketch g and f from Item 3 on the same axes below.



5. What is the domain and range of g?
The domain of g is $x \geq 3$. The range of g is $y \geq 0$.

Multiple transformations can be applied to the basic function to create a new function.

6. Describe the transformations of $f(x) = \sqrt{x}$ that result in the functions listed below.

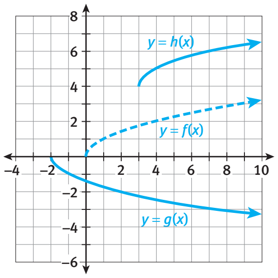
a. $g(x) = -\sqrt{x + 2}$
The graph of f reflected across the x-axis and translated 2 units to the left.

b. $h(x) = \sqrt{x - 3} + 4$
The graph of f translated 3 units right and 4 units up.

SUGGESTED LEARNING STRATEGIES: Predict and Confirm, Create Representations, Group Presentation

My Notes

7. Sketch the graph of each function in Item 6. Then state the domain and range for each function. Use a calculator to check your results.



Domain of $g: x \geq -2$, Range of $g: y \leq 0$.
Domain of $h: x \geq 3$, Range of $h: y \geq 4$.

8. Without graphing, determine the domain and range of $f(x) = \sqrt{x+5} - 1$.
Domain of $f: x \geq -5$, Range of $f: y \geq -1$.

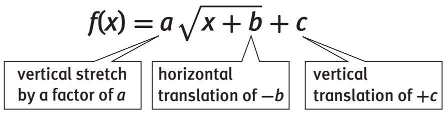
ACTIVITY 5.2 *Continued*

7 Predict and Confirm, Create Representations, Debriefing, Group Presentation Students should graph both the parent function and the transformed functions on the same axes.

8 Some students may need to graph the function before stating the domain and range. For all students, see the visual representation connecting transformation definitions and function notation below.

Connecting Transformation Definitions and Function Notation

At this point, the definitions of various transformations should be very familiar to students. However, some students may still have difficulty recognizing some transformations with a new function. A visual cue like the one shown below may be helpful. Post this after students finish Item 8.



9-10 Work Backward Reinforce the meaning of points on the graph of the hull speed function (length at waterline, hull speed). Students should locate the point where $x = 24$ on the graph to find the speed of a boat with a 24-ft length at the waterline. Students should locate the point where $y = 6$ on the graph to find the length at the waterline of a boat with a hull speed of 6 knots.

11 Create Representations, Think/Pair/Share, Debriefing Students will solve this equation at the end of Day 2 of this activity.

Suggested Assignment

CHECK YOUR UNDERSTANDING
p. 282, #1–6

UNIT 5 PRACTICE
p. 325, #6–9

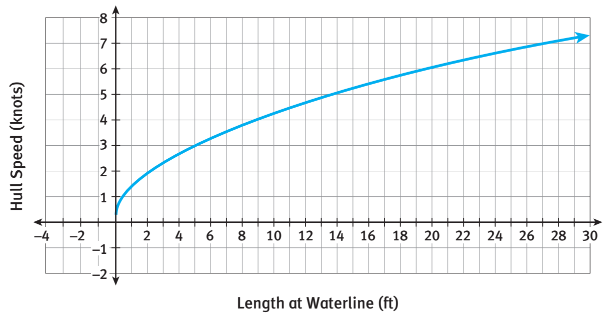
Paragraph and Steps
Note Taking, Interactive Word Wall, Vocabulary Organizer

SUGGESTED LEARNING STRATEGIES: Work Backward, Guess and Check, Create Representations, Think/Pair/Share, Note Taking, Interactive Word Wall, Vocabulary Organizer

My Notes

The graph of the hull speed of a sailboat H is shown below.

Hull Speed



9. Use the graph to estimate the hull speed of a sailboat that is 24 ft long at the waterline.
Hull speed is about 6.5 knots.

10. Use the graph to estimate the length at the waterline of a sailboat whose hull speed is 6 knots.
The boat would be about 20 feet at the waterline.

11. Write an equation that could be used to determine the length at the waterline of a sailboat with a hull speed of 6 knots.
$1.34\sqrt{x} = 6$

MATH TIP

An **extraneous solution** can be introduced when you square both sides of an equation to eliminate the square root. The resulting equation may not be equivalent to the original for all values of the variable.

To solve **square root equations**, follow these steps.

Step 1: *Isolate the radical term.*
Step 2: *Square both sides of the equation.*
Step 3: *Solve for the unknown(s).*
Step 4: *Check for **extraneous solutions**.*

SUGGESTED LEARNING STRATEGIES: Note Taking, Identify a Subtask, Guess and Check, Group Presentation

My Notes

EXAMPLE 1

Solve the equation $\sqrt{x-3}+4=9$.

$\sqrt{x-3}+4=9$

Step 1: *Isolate the radical.* $\sqrt{x-3}=5$

Step 2: *Square both sides.* $(\sqrt{x-3})^2=(5)^2$

Step 3: *Solve the equation.* $x-3=25$

$x=28$

Check the solution.

Step 4: *Substitute 28 into the original equation.* $\sqrt{28-3}+4 \mathrel{?} 9$

$5+4=9$

EXAMPLE 2

Solve the equation $x=\sqrt{x+1}+5$.

Step 1: *Isolate the radical.* $x=\sqrt{x+1}+5$

$x-5=\sqrt{x+1}$

Step 2: *Square both sides.* $(x-5)^2=(\sqrt{x+1})^2$

$x^2-10x+25=x+1$

Step 3: *Solve for x.* $x^2-11x+24=0$

$(x-3)(x-8)=0$

possible solutions $x=3, 8$

Check the possible solutions.

Step 4: $3 \mathrel{?} \sqrt{3+1}+5$ $\quad$ $8 \mathrel{?} \sqrt{8+1}+5$

$3\neq 2+5$ $\quad$ $8=3+5$

Only $x=8$ is a solution.

TRY THESE A

Solve each equation.

a. $2-\sqrt{x+1}=-5$

$x=48$

b. $\sqrt{x+4}=x-8$

$x=12$. $x=5$ is an extraneous solution.

MINI-LESSON: Square Root Equations with More Than One Radical

For students who want to extend their learning, illustrate how to solve equations with more than one square root. These equations will come up again in Unit 7 when deriving equations for conics.

Solve: $\sqrt{x-3}=4-\sqrt{x+5}$

$$(\sqrt{x-3})^2=(4-\sqrt{x+5})^2$$

$$x-3=16-8\sqrt{x+5}+x+5$$

$$3=\sqrt{x+5}$$

$$x=4$$

ACTIVITY 5.2 *Continued*

TEACHER TO TEACHER In Item 11, students wrote a square root equation. In this section, they learn the procedures for solving these equations.

Have students copy the examples into their notebooks as you work through them. Alternately, have students study these examples, then hold a whole-class discussion asking questions such as the following:

- Why do you square both sides of the equation to solve a square root equation?
- Why do you isolate the radical term? What does the word *isolate* mean?
- How does the order of operations help you identify which steps to do first when solving a radical equation?
- How does squaring both sides when there is a linear term create possible extraneous solutions?

EXAMPLES 1 AND 2

Note Taking Be sure that students don't make the mistake of saying $\sqrt{a+b}=\sqrt{a}+\sqrt{b}$. They may confuse addition of radicals with multiplication of radicals. In multiplication, it is true that $\sqrt{ab}=\sqrt{a}\cdot\sqrt{b}$ when both a and $b>0$. However, the same assumption is not true for addition of radicals.

TRY THESE A Identify a Subtask, Group Presentation, Debriefing Have students work these problems after they have understood the examples provided in the text. Use group presentations to debrief.

12 Students have an opportunity to return to the opening context and solve the equation written in Item 11.

13 Marking the Text, Predict and Confirm, Create Representations, Quickwrite, Debriefing This brief application problem will assess students' ability to read the problem closely. Be sure that students use 24 ft in their equation, the length at the waterline, not 27 ft. This Item also provides an opportunity to assess communication skills. Some students may also wish to solve the equation graphically.

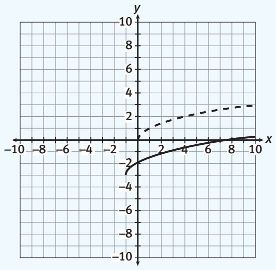
Suggested Assignment

CHECK YOUR UNDERSTANDING
p. 282, #7–11

UNIT 5 PRACTICE
p. 325, #10–12

CHECK YOUR UNDERSTANDING

1. Vertical stretch by a factor of 2, Horizontal translation 3 units to the right. Domain: $x \geq 3$, Range: $y \geq 0$.
2. Reflection across x-axis, Vertical translation 4 units up, Horizontal translation 1 unit to the left. Domain: $x \geq -1$. Range: $y \leq 4$.
3. Vertical stretch by a factor of 3, Horizontal translation of 5 to the right, vertical translation of 1 down. Domain: $x \geq 5$. Range: $y \geq -1$.
4. Parent function shown with dotted line.



5. and 6. See right side.
7. $x = 17$
8. $x = 5$, $x = 10$ is an extraneous solution.
9. $x = 100$
10. no solution
11. Answers will vary.

ACTIVITY 5.2 *continued*

Graphing Square Root Functions
Go, Boat, Go!

SUGGESTED LEARNING STRATEGIES: Marking the Text, Predict and Confirm, Create Representations, Quickwrite

My Notes

12. Solve the hull speed equation you wrote in Item 11.
$x = 20.05$ feet.

13. Maggie claims that her 27-foot sailboat *My Hero* has a hull speed of 7 knots. The length of her boat at the waterline is 24 ft. Is this claim reasonable? Explain why or why not.
Solve the equation $1.34\sqrt{x} = 7$. The solution is $x = 27.3$ feet. The length at the waterline is only 24 feet so this is not a reasonable claim.

CHECK YOUR UNDERSTANDING

Write your answers on notebook paper or grid paper. Show your work.

Describe each function as a transformation of $f(x) = \sqrt{x}$. State the domain and range.

1. $f(x) = 2\sqrt{x - 3}$
2. $f(x) = 4 - \sqrt{x + 1}$
3. $f(x) = 3\sqrt{x - 5} - 1$

Graph each function, using your knowledge of transformations.

4. $f(x) = \sqrt{x + 1} - 3$
5. $f(x) = -3\sqrt{x} + 1$
6. $f(x) = 1 - \sqrt{x - 2}$

Solve for x.

7. $\sqrt{x - 1} = 4$
8. $x + \sqrt{x - 1} = 7$
9. $2 + \sqrt{x} = 12$
10. $\sqrt{x + 4} + 7 = 3$
11. MATHEMATICAL REFLECTION What have you learned about graphing radical functions and solving radical equations in this activity?

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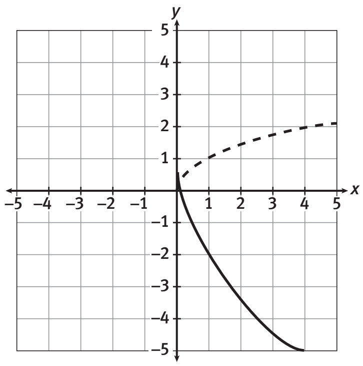
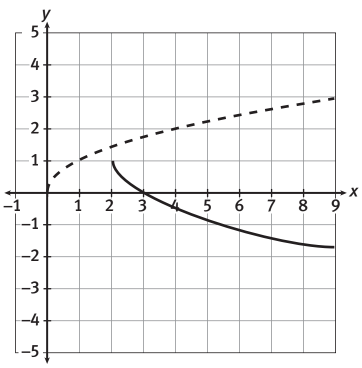
5. Parent function shown with dotted line.



6. Parent function shown with dotted line.



Rational Exponents and Radical Expressions
A Mighty Wind

ACTIVITY 5.3

SUGGESTED LEARNING STRATEGIES: Shared Reading, Summarize/Paraphrase/Retell, Marking the Text, Note Taking, Think/Pair/Share

My Notes

In 1805, Sir Francis Beaufort, a British admiral, devised a 13-point scale for measuring wind force based on how a ship's sails move in the wind. The scale is widely used by sailors even today to interpret the weather at sea and has also been adapted for use on land. It was not until the early 1900s that the Beaufort scale was related to an average wind velocity v in miles per hour using the equation $v = 1.87B^{\frac{3}{2}}$ where B is the Beaufort scale number.

1. The weather service issued a small-craft advisory with force 6 winds. Determine the wind speed predicted by the equation above, using a calculator.
The wind speed for force 6 wind is approximately 27.5 mph.

The definition below relates a **rational exponent** to a radical expression. You can use this definition to evaluate expressions without a calculator.

Definition of Rational Exponents

$$a^{\frac{1}{n}} = \sqrt[n]{a} \text{ for } a > 0 \text{ and integer } n$$

$$a^{\frac{m}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m \text{ for } a > 0 \text{ and integers } m \text{ and } n$$

2. Without using a calculator, find the wind velocity predicted by the equation $v = 1.87B^{\frac{3}{2}}$ for gale force winds ($B = 9$). Verify your answer, using a calculator.
The wind speed is 50.49 mph.

The definition of rational exponents is also useful for simplifying expressions.

EXAMPLE 1

Simplify each expression, using the definition of rational exponents.

a. $81^{\frac{1}{2}} = \sqrt{81} = 9$

b. $9^{\frac{3}{2}} = (\sqrt{9})^3 = 3^3 = 27$

c. $16^{\frac{3}{4}} = \sqrt[4]{16}^3 = 2^3 = 8$

TRY THESE A

Simplify each expression, using the definition of rational exponents. Write your answers on a separate sheet of paper. Show your work.

a. $81^{\frac{3}{4}}$ $81^{\frac{3}{4}} = \sqrt[4]{81}^3 = 3^3 = 27$ **b.** $27^{\frac{2}{3}}$ 9 **c.** $100^{\frac{5}{2}}$ 100,000

CONNECT TO METEOROLOGY

The Beaufort wind scale for use at sea is shown below.

Force	Description
0	calm
1–3	light/gentle breeze
4–5	moderate/brisk breeze
6–7	strong winds, small-craft advisory
8–9	gale force winds
10–11	storm force winds, wind damage
12	hurricane force winds

ACADEMIC VOCABULARY

rational exponent

MATH TIP

$\sqrt[n]{a}$ is called the n^{th} root of the radicand a. For square roots, the index n is not written.

The positive square root of 49 is $\sqrt{49} = 7$.

The cube root of 8 is $\sqrt[3]{8} = 2$.

ACTIVITY 5.3 Guided

Rational Exponents and Radical Expressions

Activity Focus

- Rational and real exponents
- Properties of exponents
- Simplifying radical and rational exponent expressions
- Power functions $f(x) = x^n$, where n is a rational number
- Solving equations and inequalities involving power functions with rational exponents

Materials

- Graphing calculator

Chunking the Activity

#1–2	Example 5–
Example 1–	Try These F
Try These A	#7–8
Example 2–	#9–11
Try These B	#12
Example 3–	#13–14
Try These C	#15
Example 4–	#16
Try These D	
#3	
#4	
#5–Try These E	

TEACHER TO TEACHER This activity assumes students are familiar with both the properties of exponents applied to integer exponents and the techniques to simplify radical expressions. Review these topics if necessary.

First Paragraph 1 Shared Reading, Summarize/Paraphrase/Retell, Marking the Text

Paragraph and Definition box Note Taking

2 Think/Pair/Share, Debriefing

TRY THESE A Think/Pair/Share, Debriefing

Connection to Meteorology and the Beaufort Scale

Search the internet for Beaufort Wind Scale and additional resources about the classification and measurement of sea conditions and wind speed.

Force	Wind (knots)	Description	Force	Wind (knots)	Description
0–1	0–3	calm, light air	7	28–33	near gale
2	4–6	light breeze	8	34–40	gale
3	7–10	gentle breeze	9	41–47	strong gale
4	11–16	moderate breeze	10	48–55	storm
5	17–21	fresh breeze	11	56–63	violent storm
6	22–27	strong breeze	12	64+	hurricane

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EXAMPLE 2 Note Taking, Discussion Group Students extend the laws of exponents to rational and real numbers. Students will not be able to expand expressions any longer to solve problems. They will need to add, subtract, or multiply exponents to obtain the solutions.

TRY THESE B Think/Pair/Share, Simplify the Problem, Debriefing

EXAMPLE 3 Note Taking, Discussion Group Students are asked to simplify variable expressions. All variables are assumed to be positive.

TRY THESE C Think/Pair/Share, Simplify the Problem, Group Presentation, Debriefing

TEACHER TO TEACHER You may recall that the square root $\sqrt{x}$ is treated as a function with domain $x \geq 0$. It is in this sense that $\sqrt{x^2} = |x|$ for all values of x. Notice, for example, $\sqrt{(-3)^2} = \sqrt{9} = 3 = |-3|$. Sometimes the square root function is confused with the inverse relation to the quadratic function $y = x^2$. This inverse relation is $x = y^2$ and has two "branches," a positive branch called the principal square root, and a negative branch

Suggested Assignment

CHECK YOUR UNDERSTANDING
p. 290, #1–6

UNIT 5 PRACTICE
p. 325, #13–15

SUGGESTED LEARNING STRATEGIES: Think/Pair/Share, Simplify the Problem, Note Taking, Discussion Group, Group Presentation

My Notes

MATH TIP

The *rules of exponents* given $a > 0$, $b > 0$, and real numbers m and n are listed below.

$a^m \cdot a^n = a^{m+n}$

$\frac{a^m}{a^n} = a^{m-n}$

$(a^m)^n = a^{m \cdot n}$

$\frac{1}{a^m} = a^{-m}$

$a^m \cdot b^m = (ab)^m$

$\frac{a^m}{b^m} = \left(\frac{a}{b}\right)^m$

The rules of exponents that you learned in previous mathematics classes apply to both rational and real number exponents.

EXAMPLE 2

Simplify each expression, using the properties of exponents.

a. $4^{\frac{5}{2}} \cdot 4^{\frac{1}{2}} = 4^{\frac{6}{2}} = 4^3 = 64$

b. $\left(3^{\frac{1}{3}}\right)^4 = 3^{\frac{4}{3}} = \sqrt[3]{3^4} = \sqrt[3]{3^3 \cdot 3} = 3\sqrt[3]{3}$

c. $\frac{8^{\frac{1}{2}}}{2^{\frac{1}{2}}} = \left(\frac{8}{2}\right)^{\frac{1}{2}} = 4^{\frac{1}{2}} = 2$

d. $(2^{\pi})^2 = 2^{2\pi} \approx 2^{6.28} \approx 77.7$

TRY THESE B

Simplify each expression using the properties of exponents. Write your answers in the My Notes space. Show your work.

a. $(9^3)^{\frac{1}{2}}$ $9^{\frac{3}{2}} = \sqrt{9^3} = 3^3 = 27$

b. $18^{\frac{3}{2}} \cdot 18$ $18^{\frac{5}{2}} = \sqrt{18^5} = \sqrt{18^4 \cdot 9 \cdot 2} = 18^2 \cdot 3\sqrt{2} = 972\sqrt{2}$

c. $3^{\sqrt{2}} \cdot 2^{\sqrt{2}}$ $6^{\sqrt{2}} \approx 12.6$

Variable expressions can also be simplified by using the properties of exponents.

EXAMPLE 3

Simplify $x^{\frac{3}{2}} \cdot y^{\frac{5}{2}}$ and write in radical form. Assume $x > 0$ and $y > 0$.

	Original expression	$x^{\frac{3}{2}} \cdot y^{\frac{5}{2}}$
Step 1:	*Write using definition of rational exponents.*	$\sqrt{x^3} \cdot \sqrt{y^5}$
Step 2:	*Multiply like radicals.*	$\sqrt{x^3y^5}$
Step 3:	*Factor perfect squares.*	$\sqrt{x^2y^4\,xy}$
Step 4:	*Simplify perfect squares.*	$xy^2\sqrt{xy}$

TRY THESE C

Simplify each expression and write in radical form. Assume all variables are greater than 0.

a. $\frac{x^{\frac{4}{3}} \cdot y^{\frac{5}{3}}}{x^{\frac{1}{3}}}$ $xy^{\frac{5}{3}} = xy \cdot y^{\frac{2}{3}} = xy\sqrt[3]{y^2}$

b. $\left(x^{\frac{1}{3}}\right)^{13}$ $x^{\frac{13}{3}} = x^4\, x^{\frac{1}{3}} = x^4\sqrt[3]{x}$

SUGGESTED LEARNING STRATEGIES: Activating Prior Knowledge, Think/Pair/Share, Note Taking, Discussion Group, Quickwrite, Marking the Text, Summarize/Paraphrase/Retell, Create Representations, Look for a Pattern

You can also solve equations by using the properties of exponents.

EXAMPLE 4

Solve the equation $x^{\frac{2}{3}} = 36$. Assume $x > 0$.

	Original equation	$x^{\frac{2}{3}} = 36$
Step 1:	*Raise both sides to a power that makes the exponent on x equal to 1*	$\left(x^{\frac{2}{3}}\right)^{\frac{3}{2}} = 36^{\frac{3}{2}}$
Step 2:	*Simplify using the definition of rational exponents.*	$x = (\sqrt{36})^3$ $x = 6^3$ $x = 216$

TRY THESE D

Solve each equation. Assume $x > 0$.

a. $x^{\frac{1}{4}} = 3$ 81

b. $x^{\frac{5}{2}} = 32$ 4

c. $2x^3 = 54$ 3

3. If x could be any real number, not just a positive one, would there be any other solutions to the equation shown in Example 4 above?
Answers will vary. Sample response: The number −216 could also be a solution because after you take the cube root of −216 you would square the number resulting in a positive 36.

4. Suppose the wind speed is measured at 30 mph.

a. Write an equation that could be solved to find the Beaufort number associated with this wind speed.
$1.87\ B^{\frac{3}{2}} = 30$

b. Solve the equation for B.
$B^{\frac{3}{2}} = 30 \div 1.87 \approx 16.043$; $B^3 \approx 257.38$; $B \approx 6.36$. Round to the nearest scale value of 6.

c. Use the table on the first page of the activity to interpret the wind speed.
The winds are strong enough that a small craft advisory would be issued.

My Notes

MATH TIP

Use the Power of a Power Property to isolate the variable so that it has an exponent of 1.

Example: $(x^5)^{\frac{1}{5}} = x$

CONNECT TO AP

In AP Calculus, you will work extensively with fractional and negative exponents.

EXAMPLE 4 Activating Prior Knowledge, Think/Pair/Share, Note Taking Ask students to recall how they solve equations in general. They may say something like *get the variable all by itself on one side*. Then ask them to explain to a partner how this is done in the solution to the example.

TRY THESE D Think/Pair/Share

3 Discussion Group, Quickwrite Students begin to consider why the equation in the example has more than one solution. Their understanding will increase when they explore graphs of power functions in the last portion of this activity.

4 Marking the Text, Summarize/Paraphrase/Retell, Create Representations This Item revisits the context presented at the beginning of this lesson. Students will need at least a scientific calculator to get a decimal approximation for the solution.

CONNECT TO AP

Students will work extensively with fractional exponents in AP Calculus. They need to rewrite radical expressions so they have fractional exponents. They will also frequently rewrite expressions to contain negative exponents as well. Both of these rewrites are needed to find the derivative of certain functions.

For example the function $f(x) = \dfrac{1}{\sqrt{x-2}}$ would be re-written as $f(x) = (x-2)^{-\frac{1}{2}}$ prior to finding the derivative.

First Paragraph Activating Prior Knowledge, Note Taking

5 Look for a Pattern, Create Representations Make sure that students understand that the square root $\sqrt{x}$ is defined only for $x \geq 0$, and by definition of the meaning of square root, it follows that $\sqrt{x} \cdot \sqrt{x} = x$.

TRY THESE E Think/Pair/Share, Debriefing Carefully debrief these problems to make sure students are not making careless errors. The Math Tip is there to remind students about special binomial products used in parts b and c.

Paragraph and Explanation Box Note Taking, Interactive Word Wall, Vocabulary Organizer

EXAMPLE 5 Note Taking,

6 Look for a Pattern, Discussion Group, Debriefing This Item is intended to stimulate student dialogue regarding the process of rationalizing. Rather that think of rationalizing as a procedure that applies to radicals, students should see that multiplying by 1 in a variety of forms is a useful technique for simplifying a wide variety of expressions.

Rational Exponents and Radical Expressions
A Mighty Wind

SUGGESTED LEARNING STRATEGIES: Activating Prior Knowledge, Note Taking, Look for a Pattern, Create Representations, Think/Pair/Share, Interactive Word Wall, Vocabulary Organizer

My Notes

The properties of algebra that you used to add, subtract, and multiply polynomials extend to radical expressions.

Multiply Two Binomials

$$(x - 3)(x + 2) = x^2 + 2x - 3x - 6$$
$$= x^2 - x - 6$$

5. Use the problem above as a model to multiply $(\sqrt{x} - 3)(\sqrt{x} + 2)$.

$(\sqrt{x} - 3)(\sqrt{x} + 2)$

$x - 3\sqrt{x} + 2\sqrt{x} - 6$

$x - \sqrt{x} - 6$

MATH TIP

Difference of two squares
$(a + b)(a - b) = a^2 - b^2$

Perfect square trinomials
$(a \pm b)^2 = a^2 \pm 2ab + b^2$

TRY THESE E

Simplify each expression.

a. $(\sqrt{x} + 5)(\sqrt{x} - 3)$ $\quad x + 2\sqrt{x} - 15$

b. $(3 + \sqrt{5})^2$ $\quad 14 + 6\sqrt{5}$

c. $(\sqrt{3} + 1)(\sqrt{3} - 1)$ $\quad 2$

Radical expressions are often written in simplest form.

A radical expression with *n*th roots is in **simplest form** when:
- the radicand contains no perfect n^{th} power factors other than 1
- there are no fractions in the radicand
- there are no radicals in the denominator of a fraction

The denominator of any radical expression can be rationalized by multiplying by the number 1.

MATH TIP

The binomial expressions $a + b$ and $a - b$ are **conjugates** of each other.

EXAMPLE 5

Simplify each expression.

a. $\dfrac{2}{\sqrt{5}} = \dfrac{2}{\sqrt{5}} \cdot \dfrac{\sqrt{5}}{\sqrt{5}} = \dfrac{2\sqrt{5}}{(\sqrt{5})^2} = \dfrac{2\sqrt{5}}{5}$

b. $\sqrt[3]{\dfrac{8}{5}} = \dfrac{\sqrt[3]{8}}{\sqrt[3]{5}} = \dfrac{2}{\sqrt[3]{5}} \cdot \dfrac{\sqrt[3]{5^2}}{\sqrt[3]{5^2}} = \dfrac{2\sqrt[3]{25}}{\sqrt[3]{5^3}} = \dfrac{2\sqrt[3]{25}}{5}$

c. $\dfrac{3}{1 + \sqrt{5}} = \dfrac{3}{1 + \sqrt{5}} \cdot \dfrac{1 - \sqrt{5}}{1 - \sqrt{5}} = \dfrac{3(1 - \sqrt{5})}{1 - \sqrt{5} + \sqrt{5} - (\sqrt{5})^2}$

$= \dfrac{3 - 3\sqrt{5}}{1 - 5} = \dfrac{3 - 3\sqrt{5}}{-4} = -\dfrac{3}{4} + \dfrac{3\sqrt{5}}{4}$

6. In each example above, circle the factor equal to the number 1.

SUGGESTED LEARNING STRATEGIES: Simplify the Problem, Group Presentation, Interactive Word Wall, Vocabulary Organizer, Activating Prior Knowledge, Think/Pair/Share, Summarize/Paraphrase/Retell, Create Representations, Look for a Pattern, Quickwrite

My Notes

TRY THESE F

Simplify each expression. Write your answers in the My Notes space. Show your work.

a. $\dfrac{5}{\sqrt[3]{10}}$ **b.** $\dfrac{1+\sqrt{2}}{\sqrt{3}}$ **c.** $\dfrac{4}{\sqrt{5}-2}$ **d.** $\dfrac{2+\sqrt{3}}{3+\sqrt{7}}$

a. $\dfrac{5}{\sqrt[3]{10}} \cdot \dfrac{\sqrt[3]{10^2}}{\sqrt[3]{10^2}} = \dfrac{5\sqrt[3]{100}}{10} = \dfrac{\sqrt[3]{100}}{2}$

b. $\dfrac{\sqrt{3}}{3} + \dfrac{\sqrt{6}}{3}$

c. $4\sqrt{5} + 8$

d. $3 - \sqrt{7} + \dfrac{3\sqrt{3}}{2} - \dfrac{\sqrt{21}}{2}$

In a previous activity, you explored the graphs of square root functions. The square root function is an example of a *power function*. A function of the form $f(x) = x^n$ for any real number n is a **power function**.

ACADEMIC VOCABULARY

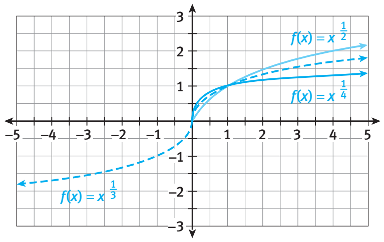
power function

7. Write the parent square root function as a power function.
$f(x) = x^{\frac{1}{2}}$

8. List some other power functions that you have studied previously.
Answers will vary. Sample response $f(x) = x$, $f(x) = x^2$, $f(x) = x^3$.

Other power functions with rational exponents have some interesting properties.

9. Graph the parent square root function and the power functions $f(x) = x^{\frac{1}{3}}$ and $f(x) = x^{\frac{1}{4}}$ on a graphing calculator. Then sketch the result on the grid below.



10. How do the graphs of these functions compare?
Answers will vary. Sample response: In the first quadrant, the functions $f(x) = x^{\frac{1}{3}}$ and $f(x) = x^{\frac{1}{4}}$ are similar to the parent function $f(x) = x^{\frac{1}{2}}$ but shrunk vertically. They all pass through (0, 0) and (1, 1). The domain of the function $f(x) = x^{\frac{1}{3}}$ is all real numbers, since it is reflected through the origin.

9 Create Representations Student sketches should label the intersection points (0, 0) and (1, 1), and each graph should be labeled clearly.

10 Summarize/Paraphrase/Retell, Look for a Pattern, Quickwrite Emphasize proper vocabulary on this Item.

ACTIVITY 5.3 Continued

TRY THESE F Simplify the Problem, Group Presentation, Debriefing Here are some additional questions to ask students during debriefing:

- How did multiplying by the conjugate in the last example eliminate the radical?
- How does the process of rationalizing with an nth root other than $n = 2$ differ from rationalizing with square roots?
- Why would you rewrite the fractions as two separate radicals first when simplifying?
- Do you think you could "rationalize" the numerator of an expression?

AP Connection

The answer to this last question for Try These F above is yes, and rationalizing numerators is occasionally used when working with certain types of limits in advanced mathematics classes.

Suggested Assignment

CHECK YOUR UNDERSTANDING
p. 290, #7–11

UNIT 5 PRACTICE
p. 325–326, #16–19

TEACHER TO TEACHER Students explore graphs of power functions with rational exponents. They solve equations and inequalities by finding intersection points of graphs. Students will need to use a graphing calculator.

First Paragraph Interactive Word Wall, Vocabulary Organizer

7-8 Activating Prior Knowledge, Think/Pair/Share, Debriefing These Items connect the definition of power function to functions students have already studied.

11 (*Continued*) Quickwrite, Debriefing
Students previously studied complex numbers. They need to understand that even roots of negative values are undefined in the real number system.

12 Predict and Confirm, Look for a Pattern This Item is an opportunity to assess whether or not students can generalize the patterns they explored in Items 9–11.

13-14 Summarize/Paraphrase/Retell, Create Representations, Look for a Pattern, Quickwrite, Debriefing These Items are similar to Items 9–10. They will help students to understand more fully the ideas explored in Item 3.

My Notes

SUGGESTED LEARNING STRATEGIES: Quickwrite, Predict and Confirm, Look for a Pattern, Summarize/Paraphrase/Retell, Create Representations, Quickwrite

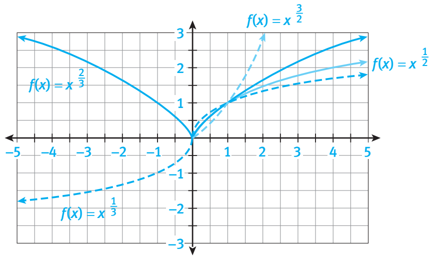
11. Why is the function $f(x) = x^{\frac{1}{3}}$ defined for all real values of x while the other two functions in Item 9 are not?
The cube root of a negative number is a real number. The n^{th} root of a number when n is even is not a real number.

12. Without graphing, predict the domain of each function.

a. $f(x) = x^{\frac{1}{5}}$
all real numbers

b. $f(x) = x^{\frac{1}{6}}$
$x \geq 0$

13. Graph the parent square root function and the power functions $f(x) = x^{\frac{3}{2}}, f(x) = x^{\frac{1}{3}}$, and $f(x) = x^{\frac{2}{3}}$ on a graphing calculator. Then sketch the result on the grid below.



14. How do the graphs of these functions compare?
Answers will vary. Sample response. They all contain the points (0, 0) and (1, 1). The graphs with powers less than one are "curved down" for $x > 0$. The functions with exponents with odd denominators are defined for all real numbers. The functions with exponents with even denominators are defined for $x \geq 0$.

SUGGESTED LEARNING STRATEGIES: Create Representations, Look for a Pattern, Discussion Group, Quickwrite, Marking the Text, Activating Prior Knowledge, Note Taking, Group Presentation

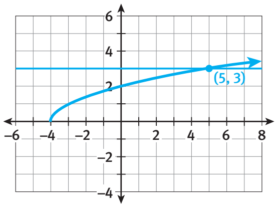
My Notes

15. How do the graphs of the functions in Item 13 help to explain the fact that an equation like $x^{\frac{3}{2}} = 8$ will only have 1 solution but an equation like $x^{\frac{2}{3}} = 8$ will have 2 solutions?

Answers will vary. Sample response: If you draw a horizontal line through the graph for some number greater than 0, it will intersect $f(x) = x^{\frac{2}{3}}$ twice but it will only intersect $f(x) = x^{\frac{3}{2}}$ once. This means there will be two x-values that satisfy the one equation and only one x-value that satisfies the second one.

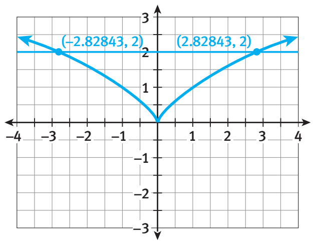
16. Solve each equation and inequality, using a graphing calculator. Enter the left side as one function and the right side as another function. Solve graphically or algebraically.

a. $2\sqrt{x+4} = 6$



The solution is $x = 5$.

b. $x^{\frac{2}{3}} = 2$



The solutions are approximately −2.83 and 2.83.

ACTIVITY 5.3 *Continued*

15 Create Representations, Look for a Pattern, Discussion Group, Quickwrite, Debriefing This Item uses the graph of the function to help students understand why there are two solutions to certain equations of the form $x^n = k$ where n is a rational number, while others have one (or possibly no) solutions.

16 Marking the Text, Create Representations, Activating Prior Knowledge, Note Taking, Group Presentation, Debriefing If students are not familiar with using a graphing calculator to solve an equation or inequality, provide some extra assistance on these problems.

Take time to explore the Math Tip with your class and have students take notes if solving inequalities graphically is unfamiliar to them. Ask them to recall how they did this in previous units.

If students are familiar with using technology in this way, consider assigning each group a different problem and then use group presentations to debrief student work.

ACTIVITY 5.3 *Continued*

16 (*continued*) Marking the Text, Create Representations, Group Presentation, Debriefing As a final debriefing activity, have the class create a graphic organizer summarizing what they learned about power functions and how they can use their graphs to solve equations and inequalities.

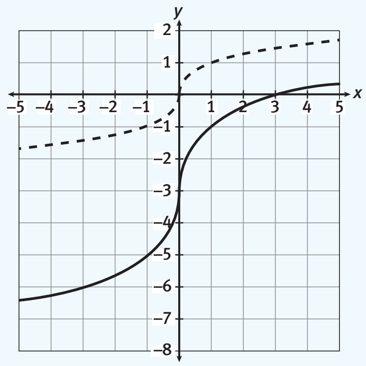
Suggested Assignment

CHECK YOUR UNDERSTANDING
p. 290, #12–14

UNIT 5 PRACTICE
p. 326, #20–21

CHECK YOUR UNDERSTANDING

1. 25
2. $8\sqrt[4]{8}$
3. $\frac{5\sqrt{3}}{9}$
4. $x^4y\sqrt{xy}$
5. $xy\sqrt[3]{x}$
6. $\frac{x\sqrt[3]{x}}{y^2}$
7. $x = 1000$
8. $x = 3$
9. $x - 16\sqrt{x} + 64$
10. $\sqrt{x^2 - y^2}$
11. $\frac{\sqrt[3]{4}}{2} + 1$
12. g is the graph of f stretched vertically by a factor of 2 and translated down 3 units.



13a. $x > 47$

b. $x = 8$ or -8

14. Answers may vary.

SUGGESTED LEARNING STRATEGIES: Marking the Text, Create Representations, Group Presentation

My Notes

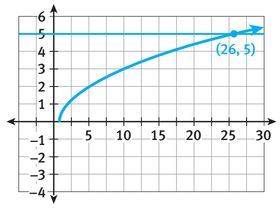
MATH TIP

If $f(x) > k$, the solution interval will be those x-values in the function domain where $f(x)$ is *above* the line $y = k$.

If $f(x) < k$, the solution interval will be those x-values in the function domain where $f(x)$ is *below* the line $y = k$.

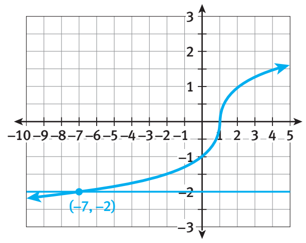
16. (*continued*)

c. $\sqrt{x - 1} > 5$



The solution is $x > 26$.

d. $\sqrt[3]{x - 1} + 4 \le 2$



The solution is $x \le -7$.

CHECK YOUR UNDERSTANDING

Write your answers on notebook paper or grid paper. Show your work.

Simplify each expression.

1. $125^{\frac{2}{3}}$ **2.** $8^{\frac{3}{4}} \cdot 8^{\frac{1}{2}}$

3. $\left(\frac{27}{25}\right)^{-\frac{1}{2}}$ **4.** $\left(x^{\frac{3}{2}}y^{\frac{1}{2}}\right)^3$

5. $\sqrt{\sqrt[3]{x^8y^6}}$ **6.** $\left(\frac{x^4}{y^6}\right)^{\frac{1}{3}}$

Solve each equation. Assume $x > 0$.

7. $5x^{\frac{1}{3}} = 50$ **8.** $4x^4 - 324 = 0$

Simplify each expression.

9. $(\sqrt{x} - 8)^2$ **10.** $\sqrt{x + y} \cdot \sqrt{x - y}$

11. $\frac{1 + \sqrt[3]{2}}{\sqrt[3]{2}}$

12. Describe $g(x) = 2x^{\frac{1}{3}} - 3$ as a transformation of $f(x) = x^{\frac{1}{3}}$. Use this information to sketch the graph of g.

13. Solve graphically or algebraically.

a. $\sqrt{x + 2} - 3 > 4$ **b.** $3x^{\frac{2}{3}} = 12$

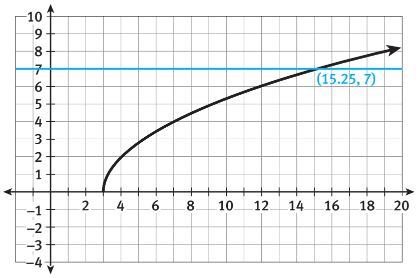
14. MATHEMATICAL REFLECTION How does the mathematics of this activity relate to what you have learned previously about simplifying expressions and solving equations?

Square Root Expressions, Equations, and Functions

Embedded Assessment 1
Use after Activity 5.3.

A MIGHTIER WIND

1. The graph of a function g is shown below.



a. Describe the graph as a transformation of $f(x) = \sqrt{x}$.

b. Write the equation for g.

c. State the domain and range of g.

d. Find the inverse of g. Be sure to include any restrictions on the domain of the inverse.

e. Use the graph or a table to solve the inequality $g(x) > 7$.

2. Solve the equation $x + \sqrt{x} = 6$.

3. Classify each statement as *sometimes*, *always*, or *never true*.

a. $2^{\frac{2}{3}} \cdot 4^{\frac{4}{3}} = 8 \cdot 2^{\frac{1}{3}}$

b. If $x > 0$, then $(\sqrt{x} \cdot \sqrt{2})^3 = x^{\frac{3}{2}}\ 2^{\frac{3}{2}}$.

c. If $x > 0$, then $(\sqrt{x} + \sqrt{2})^3 = x^{\frac{3}{2}} + 2^{\frac{3}{2}}$.

d. The only solution to $x^{\frac{2}{3}} = 4$ is $x = 8$.

Embedded Assessment 1

Assessment Focus

- Inverse functions
- Composition of functions
- Transformations of $f(x) = \sqrt{x}$
- Square root equations
- Rational exponents

Materials

- Graphing calculator

TEACHER TO TEACHER If there are time constraints, have students complete the first page in class and assign the second page as a take-home assessment or for the next day.

1 This Item focuses on transformations of the parent square root function. Students also find the inverse of the function they wrote in part (a) and then solve an inequality. If students are not using a calculator, allow additional time to complete this activity.

Answer Key

1a. It is the parent function vertically stretched by a factor of 2 and translated 3 units to the right.

b. $g(x) = 2\sqrt{x - 3}$

c. Domain of g: $x \geq 3$, Range of g: $y \geq 0$

d. $g^{-1}(x) = \left(\frac{x}{2}\right)^2 + 3$ for $x \geq 0$.

2. 4

2 This equation has an extraneous solution that students should eliminate upon checking their solution.

$$x + \sqrt{x} = 6$$
$$\sqrt{x} = 6 - x$$
$$(\sqrt{x})^2 = (6 - x)^2$$
$$x = 36 - 12x + x^2$$
$$x^2 - 13x + 36 = 0$$
$$(x - 4)(x - 9) = 0$$
$$x = 4, 9$$

check: $4 + \sqrt{4} = 6$

check: $9 + \sqrt{9} \neq 6$

3 Students may find these Items difficult, but their responses will inform their thinking. For example, a student who answers *always* to part (c) does not have a complete understanding of solving rational equations. Revisiting the second half of Activity 5.3 will help those students.

3a. always

b. always

c. never

d. never

Embedded Assessment 1

4 This Item gives students a contextual situation related to Activity 5.3. The International Tornado Intensity Scale and the Beaufort scale are related to one another by a linear equation. Students derive this equation in part (a) and then use it to answer parts (b)–(d).

The equations T and B could be written using function notation. The domain of function T is all the B-values and the domain of function B is all the T-values. In part (c), the students must recall the composition property of inverses. In part (d), they will use their answer from part (b) to write v as a function of T.

As with the Beaufort scale, the wind velocity v is an approximate value. The T-scale numbers are wind speed bands that represent the potential damage caused by a tornado. For example, a T-2 tornado has wind velocities that range from 73 to 92 mph and would cause moderate damage. More information can be found at the Tornado Storm and Research Organization (TORRO) online.

4a. $T = 0.5(B - 8)$

b. $B = 2T + 8$

c. 10; the intensity of a tornado that has a force of 10

d. $v = 1.87(2T + 8)^{\frac{3}{2}}$

Differentiating Instruction

Read through Item 4 with English learners and pre-teach vocabulary related to the International Tornado Intensity Scale.

Embedded Assessment 1
Use after Activity 5.3

Square Root Expressions, Equations, and Functions

A MIGHTIER WIND

4. The International Tornado Intensity Scale (T-Scale) is widely used in Europe to describe the wind force of tornados. It was developed to extend the Beaufort scale to classify meteorological events with very high wind speeds. A TORRO force $T = 0$ is the same as the Beaufort force $B = 8$ and a TORRO force $T = 2$ is the same the Beaufort force $B = 12$.

a. Write T as a linear function of B, using the ordered pairs (8, 0) and (12, 2).

b. Use what you have learned about inverse functions to solve for B as a function of T.

c. What is the value and meaning of $T(B(10))$?

d. Use composition of functions and the equation for wind velocity $v = 1.87B^{\frac{3}{2}}$ to express v as a function of T.

	Exemplary	Proficient	Emerging
Math Knowledge #1c, e; 2, 4b, c	The student: • States both the correct domain and range of g. (1c) • Solves the inequality correctly. (1e) • Solves the equation correctly. (2) • Solves for B as a function of T correctly. (4b) • Finds the correct value for $T(B(10))$. (4c)	The student: • States either the correct domain or range, but not both. • Uses the correct method to solve the equation but makes a computational or mathematical error. • Uses the correct method to solve for the function but makes a computational or mathematical error.	The student: • States neither the correct domain nor range. • Does not solve the inequality correctly. • Does not use the correct method to solve the equation. • Does not use the correct method to solve for the function. • Does not find the correct value for $T(B(10))$.
Problem Solving #1d, 3a–d	The student: • Finds the correct inverse of g and states its correct domain. (1d) • Classifies the four statements correctly. (3a–d)	The student: • Finds the correct inverse of g, but states an incorrect domain. OR • States the correct domain for an incorrect inverse. • Classifies only three of the four statements correctly.	The student: • Finds an incorrect inverse and an incorrect domain. • Classifies at least one of the statements correctly.
Representations #1b, 4a, d	The student: • Writes a correct equation for g. (1b) • Writes a correct linear function. (4a) • Correctly expresses v as a function of T. (4d)	The student: • Writes a partially correct equation for g. • Writes an incorrect linear function. • Writes a partially correct function.	The student: • Writes an incorrect equation for g. • Writes a function that is not linear. • Writes an incorrect function.
Communication #1a, 4c	The student: • Gives a complete and accurate description of the transformation. (1a) • Gives the correct meaning of $T(B(10))$. (4c)	The student: • Gives an incomplete description of the transformation with no mathematical errors. • Gives a partially correct meaning of $T(B(10))$.	The student: • Gives a description of the transformation that contains mathematical errors. • Gives an incorrect meaning of $T(B(10))$.

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292 SpringBoard® Mathematics with Meaning™ Algebra 2

Introduction to Rational Functions
Planning a Summer Camp

ACTIVITY 5.4

SUGGESTED LEARNING STRATEGIES: Marking the Text, Summarize/Paraphrase/Retell, Create Representations, Look for a Pattern

My Notes

The finance committee of a nonprofit summer camp for children is setting the cost for a 5-day camp. The fixed cost for the camp is $2400 per day, and includes things such as rent, salaries, insurance, and equipment. An outside food services company will provide meals at a cost of $3 per camper, per meal. Campers will eat 3 meals a day.

As a nonprofit camp, the camp must cover its costs, but not make any profit. The committee must come up with a proposal for setting the fee for each camper, based on the number of campers who are expected to attend each week.

1. Initially, the committee decides to calculate camper fees based on the fixed cost of the camp alone, without meals for the campers.

a. What is the total fixed cost for the five days? $12,000

b. Complete the table below to determine the fee per camper that will guarantee the camp does not lose money.

Number of Campers	Fee per Camper
25	$480
50	$240
75	$160
100	$120
200	$60
500	$24
1000	$12
x	$\frac{12{,}000}{x}$

MATH TIP

Use the patterns you observe in the table to write an algebraic expression in the last row when there are x campers.

ACTIVITY 5.4 Investigative

Introduction to Rational Functions

Activity Focus
- Rational functions
- Domain and range
- Asymptotes
- Writing rational functions to model situations
- Graphing rational functions

Materials
- Graphing calculator

Chunking the Activity

#1–2	#8–9	#17
#3	#10–13	#18
#4–7	#14–16	#19–20

TEACHER TO TEACHER Students develop several rational function models to set a fee per camper. Students explore the characteristics of rational functions graphically and numerically. They have many opportunities to communicate their understanding and make decisions based on their work.

First Two Paragraphs Marking the Text, Summarize/Paraphrase/Retell

1 a–b Create Representations, Look for a Pattern These Items help students gain entry into the problem. As students work in their groups, check their progress to make sure that they have correctly processed the situation. Some may approach the problem by dividing $12,000 by the number of campers in attendance to complete the table. The repetition of this process should help them generalize in the last row.

1 c–d Create Representations
After students have completed their graphs, start a discussion about the domain and range of this function, emphasizing that the domain must be the positive integers. You might ask a question about the range such as, "What happens to the cost as the number of campers increases?"

2 Quickwrite, Debriefing
Be sure to emphasize the use of appropriate vocabulary here. Take time to refer students to their math notebooks for terms they should already know.

Points for discussion during a debriefing should include recognizing that the graph is discrete, decreasing, and does not have any intercepts. Students may realize that the graph does not extend below the *x*-axis, but the discussion of asymptotes can be deferred until later in this activity.

Differentiating Instruction

If students have difficulty, post a vocabulary list for describing graphs, as shown below. Students may not be familiar with all the terms in the list, but should find the list helpful nonetheless.

3 Quickwrite, Group Presentation, Debriefing
Conduct a debriefing session after students have attempted to work on this Item. Be open to all student answers. Some students may say that the minimum is 1 cent. Others will realize that in reality the camp will have a maximum capacity and that the fee cannot be less than $\frac{12{,}000}{M}$, where *M* is the greatest number of campers that can fit at the camp. The issue of rounding can be addressed at this time. In this context, it is appropriate to round up.

4 a Marking the Text

ACTIVITY 5.4 *continued*

Introduction to Rational Functions
Planning a Summer Camp

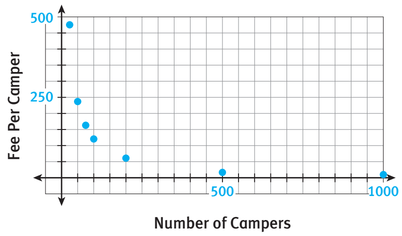
SUGGESTED LEARNING STRATEGIES: Create Representations, Quickwrite, Group Presentation, Marking the Text

My Notes

MATH TIP
You can use the values in your table to help you determine an appropriate scale for a graph.

1. (*continued*)

c. Using an appropriate scale, make a graph showing the relationship between the fee per camper and the number of campers in attendance.



d. Write an algebraic rule for the fee per camper as a function of the number of campers in attendance.

$f(x) = \frac{12{,}000}{x}$, where x is the number of campers and $f(x)$ is the fee per camper.

2. Describe the features of the graph in Item 1(c).

Answers will vary. Sample answer: The graph is discrete, decreasing, and non-negative. As x increases, the function approaches zero. As x approaches 0, the function increases.

3. Based on your work so far, is there a minimum camper fee, not counting the cost of meals? If so, what is it? Explain.

Answers will vary. Sample answer: $f(x) \to 0$ as $x \to \infty$. However, the camp may not hold even 1000 campers. If it holds 500 campers, then the minimum fee would be $\frac{12{,}000}{500} = 24$ dollars.

4. The function developed in Item 1 did not account for meals. Campers eat three meals per day at a cost of \$3 per camper per meal. The committee must determine a function that includes the cost of meals when setting the fee per camper.

a. What will be the total cost for meals per camper each week? \$45

Vocabulary for Describing Graphs

asymptotes	increasing	slope
continuous	maximum point	symmetry
decreasing	minimum point	*x*-intercepts
discontinuous	range	*y*-intercept
discrete	rate of change	zeros
domain		

Use a graphic organizer, interactive word wall, and vocabulary notebooks to help students review and organize these important terms.

SUGGESTED LEARNING STRATEGIES: Create Representations, Look for a Pattern

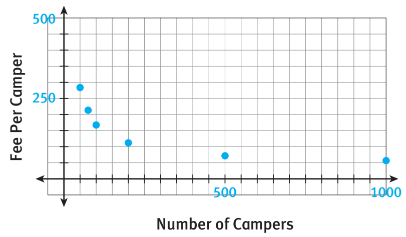
My Notes

4. (*continued*)

b. Complete the table below to determine the fee per camper that will guarantee the camp does not lose money.

Number of Campers	Fixed Cost plus the Cost of Meals	Fee per Camper
25	12000 + 45(25) = \$13,125	\$525
50	\$14,250	\$285
75	\$15,375	\$205
100	\$16,500	\$165
200	\$21,000	\$105
500	\$34,500	\$69
1000	\$57,000	\$57
x	$12{,}000 + 45x$	$\frac{12{,}000 + 45x}{x}$

c. Using an appropriate scale, make a graph showing the relationship between the fee per camper, including meals, and the number of campers.



d. Write an algebraic rule for the fee per camper, including meals, as a function of the number of children in attendance.

$f(x) = \frac{12{,}000 + 45x}{x}$ where x is the number of campers and $f(x)$ is the fee per camper.

CONNECT TO AP

Describing the behavior of rational functions as they approach horizontal and vertical asymptotes provides an introduction to a more formal study of limits that will occur in calculus.

ACTIVITY 5.4 *Continued*

4 b Create Representations Part (a) and this Item prepare students to develop a modified model that takes into account the cost of 5 days worth of meals for each camper.

4 c Create Representations Check students' graphs to make sure they are discrete points.

4 c Create Representations, Look for a Pattern Students should use their work on the table to generalize the function rule. Look for students who write their function as $f(x) = 45 + \frac{12000}{x}$. Their reasoning may be that everyone pays \$45 for meals plus their share of the fixed costs. Before moving on, make sure all groups have the correct algebraic rule for the function.

CONNECT TO AP

Rational functions are included in the list of prerequisites for AP Calculus. Students need to be able to analyze the behavior of rational functions and recognize their important features. When students describe the behavior of rational functions as the functions approach the horizontal and vertical asymptotes, they are building the conceptual foundation for a more formal study of limits that will occur in calculus.

ACTIVITY 5.4 *Continued*

5 Quickwrite

6 Quickwrite Students should recognize that as the number of children in attendance increases, the cost per camper approaches \$45, and they may give \$45 as a minimum cost. Others may suggest the minimum cost is \$45 plus 12,000 divided by the maximum number of campers based on the camp's capacity. When compared to their original response, the new minimum should reflect the \$45 per camper cost of food.

7 Quickwrite, Self Revision/ Peer Revision, Group Presentation, Debriefing

Suggested Assignment

CHECK YOUR UNDERSTANDING
p. 300, #1–2

UNIT 5 PRACTICE
p. 326, #22–24

8 a Create Representations
As you circulate around the classroom, observe how students are completing the table. Guide them if needed to understand that they will be dividing by the number of paying children, not the total number of children in attendance.

SUGGESTED LEARNING STRATEGIES: Quickwrite, Self Revision/ Peer Revision, Group Presentation, Create Representations

My Notes

5. Based on your work so far, is there a minimum camper's fee? If so, what is it? Explain your reasoning.
Answers will vary. Sample answer: Everyone needs to pay at least \$45 to cover the cost of the meals plus their share of the fixed costs found by dividing 12,000 by the number of campers.

6. How does your answer to Item 5 differ from the one you gave for Item 3?
Item 5 includes the cost of meals, so the limit changed.

7. Describe the difference between the graphs in Items 1(c) and 4(c).
Answers will vary. Sample answer: The y-values differ by 45. The graph is still decreasing, discrete and positive, but it is getting closer to 45 as x gets larger.

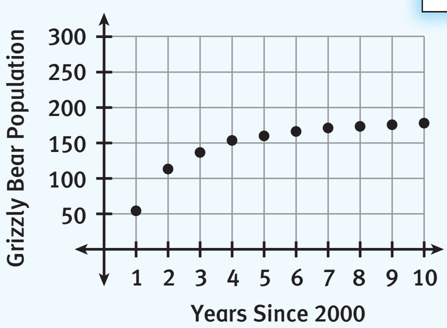
8. The committee decides to award 30 scholarships to students who otherwise could not afford the camp. These scholarships include full use of the facilities and all meals at no charge.

a. To help account for the scholarships, complete the table below.

Number of Campers	Fixed Cost plus Cost of Meals	Number of Paying Campers	Fee per Paying Camper
50	\$14,250	20	\$712.50
75	\$15,375	45	\$341.67
100	\$16,500	70	\$235.71
200	\$21,000	170	\$123.53
500	\$34,500	470	\$73.40
1000	\$57,000	970	\$58.76
x	$12{,}000 + 45x$	$x - 30$	$\frac{12{,}000 + 45x}{x - 30}$

CHECK YOUR UNDERSTANDING

1.



2. The graph is increasing from $x = 1$, discrete, and non-negative.

3. domain: counting numbers; range: all real numbers greater than or equal to 120 and less than or equal to 200

4. 160

5. The model predicts 188 grizzly bears in the year 2018.

6a. vertical $x = 3$, horizontal $y = 1$

b. x-intercept is -2. y-intercept is $-\frac{2}{3}$.

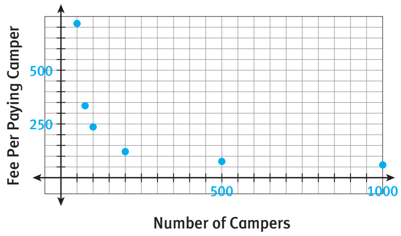
c. See page 298.

SUGGESTED LEARNING STRATEGIES: Create Representations, Quickwrite, Self Revision/Peer Revision, Group Presentation

My Notes

8. (*continued*)

b. Using an appropriate scale, make a graph showing the relationship between the fee per paying camper and the number of campers.



c. Write an algebraic model for the fee per paying camper as a function of the number of campers in attendance.

$f(x) = \frac{12{,}000 + 45x}{x - 30}$ where x is the number of campers and $f(x)$ is the fee per camper.

9. Based on your work so far, is there a minimum camper's fee? If so, what is it? Explain.

The minimum would be 45 plus 13,350 (the fixed cost plus meals for 30 scholarship campers) divided by the number of paying campers.

10. How does your answer to Item 9 differ from the one you gave for Item 5?

It takes into account the fact that the paying campers have to absorb the cost on the scholarship campers meals and their share of the fixed costs.

11. How does your graph in Item 8(b) compare to the one in Item 4(c)?

The graphs are decreasing, discrete, and approaching 45, but the x-values must be greater than 30.

ACTIVITY 5.4 *Continued*

8 b Create Representations Make sure students are graphing the total number of campers as the x-value, not the number of paying campers.

8 c Create Representations Check to see that students are dividing by $x - 30$, the number of paying children, when they write their algebraic rule.

9-10 Quickwrite The purpose of these Items is to have students investigate the end behavior of the function found in Item 8(c) over the appropriate domain. The camper's fee approaches \$45, which is the same value as in Item 5. The minimum fees approach the same value, because as the number of campers increases, the extra fee for the campers with scholarships becomes insignificant.

11 Quickwrite, Self Revision/Peer Revision, Group Presentation While students may think that the graph has shifted right 30 units, this is not correct. The graph of the function $f(x) = \frac{12{,}000 + 45(x - 30)}{x - 30}$ would be the horizontal translation of the graph of $f(x) = \frac{12{,}000 + 45x}{x}$ shifted 30 units to the right. What is true about the new function and the original one is that they have the same horizontal asymptote $y = 45$. The term *horizontal asymptote* is defined on page 299.

TEACHER TO TEACHER You can discuss rewriting the function $f(x) = \frac{12{,}000 + 45x}{x - 30}$ to make the horizontal asymptote more apparent. Using long division, the function can be rewritten as $f(x) = 45 + \frac{13{,}350}{x - 30}$. In this situation, the number 13,350 represents the fixed cost of \$12,000 plus the cost of meals for the 30 students on fee waiver, 30(45) or \$1350. As the number of paying campers increases, the amount each one pays toward the \$13,350 decreases, but each paying camper still pays \$45 for meals.

12-13 **Debriefing** Students should be able to evaluate the function they have written to determine the answer to this Item. Students should recognize that a value of –2625 for the cost per camper is not appropriate. There must be more than 30 campers with at least one paying, so the domain of the function is integers greater than 30.

14 **Work Backward** Students may solve the equation analytically or by finding the intersection of f and 80 on a graphing calculator. Make sure students subtract 30 from their solution to reflect the actual number of paying campers.

15-16 **Work Backward, Quickwrite, Debriefing** These Items emphasize inverse functions. To find the number of campers as a function of the fee per paying camper, let y equal $f(x)$, exchange y and x, and solve for y.

TEACHER TO TEACHER You can review the concept of the inverse of a function at this point by looking at the graphs of the functions with respect to the line $y = x$ in a square viewing window on the calculator and by comparing the domain and range values. Pick an ordered pair (a, b) and have students demonstrate the following: If $f(a) = b$, then $g(b) = a$, and $f(g(b)) = b$ and $g(f(a)) = a$.

17 **Prewriting, RAFT** This Item is an opportunity for students to communicate their understanding of the work they have done so far. You might assign this Item for homework.

Suggested Assignment

CHECK YOUR UNDERSTANDING
p. 300, #3–5

UNIT 5 PRACTICE
p. 326, #25–26

My Notes

SUGGESTED LEARNING STRATEGIES: Guess and Check, Work Backward, Quickwrite, Prewriting, RAFT

12. If the number of campers is 25, what is the fee per paying camper? What does your answer tell you about the limitations of this model?
$f(25) = -2625$. This answer means that the function is not valid for 30 campers or less.

13. What is the domain of the function for the fee per paying camper?
The counting numbers where $x > 30$.

14. Last year the weekly camper fee was \$80. If the camp charges the same amount and grants 30 scholarships, what is the minimum number of paying campers that must attend so the camp does not lose money?
Solve $80 = \dfrac{12{,}000 + 45x}{x - 30}$. $x = 411.429$, which rounds to 412.
The number of paying campers is $412 - 30 = 382$ campers.

15. Express the number of campers as a function of the fee for each paying camper.
Solve for x.

$$f(x) = \frac{12{,}000 + 45x}{x - 30}$$
$$y = \frac{12{,}000 + 45x}{x - 30}$$
$$y(x - 30) = 12{,}000 + 45x$$
$$xy - 30y = 12{,}000 + 45x$$
$$xy - 45x = 12{,}000 + 30y$$
$$x(y - 45) = 12{,}000 + 30y$$
$$x = \frac{12{,}000 + 30y}{y - 45}$$

or $g(x) = \dfrac{12{,}000 + 30x}{x - 45}$, where $x =$ the fee per paying camper and $g(x) =$ the number of children in attendance

16. What is the relationship between the function in Item 8(c) and the function in Item 15?
The two functions are inverses.

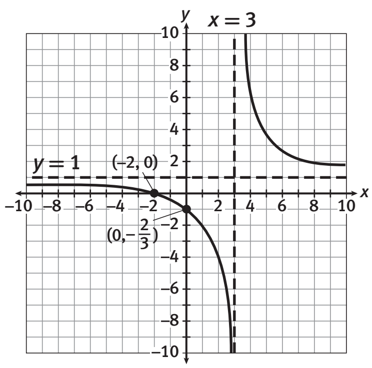
The camp can accommodate up to 300 campers, and market research indicates that campers do not want to pay more than \$200 per week. Although the camp is nonprofit, it cannot afford to lose money.

17. On a separate sheet of paper, write a proposal for setting the fee per camper. Be sure to include these items. Answers will vary.

- the proposed fee
- the minimum number of campers needed to break even
- the maximum possible income for the proposed fee
- mathematics to support your reasoning

CHECK YOUR UNDERSTANDING

6c.



SUGGESTED LEARNING STRATEGIES: Vocabulary Organizer, Interactive Word Wall, Create Representations, Note Taking

When using a function to model a situation like the fee per camper, you only use those values that make sense in the context of the situation. Items 18–20 consider the **rational function** $f(x) = \frac{12000 + 45x}{x - 30}$ over a broader range of values.

18. Graph the function on a graphing calculator, using the viewing window $[-450, 450]$ by $[-400, 400]$.

a. Use your calculator to approximate the x- and y-intercepts.
The x-intercept is approximately $(-267, 0)$. The y-intercept is approximately $(0, -400)$.

b. Find the exact values of the x- and y-intercepts, using the function. Show your work.
Solve $12{,}000 + 45x = 0$. The x-intercept is $-266\frac{2}{3}$. The y-intercept is $f(0) = \frac{12000}{-30} = -400$.

c. Name the value(s) for which the function is not defined and explain how you determined the value(s). Recall that division of a nonzero quantity by zero is *undefined*.
The function is not defined when the denominator is equal to zero. Therefore, the function is undefined for $x = 30$.

d. What is the domain of the function?
All real numbers except $x = 30$.

e. What is the range of the function?
All real numbers except $y = 45$.

If the values of function f approach some number a as the absolute value of x becomes large without bound, the line $y = a$ is called a **horizontal asymptote** of f. If the absolute value of function f increases without bound as x approaches some number b, then the line $x = b$ is a **vertical asymptote** of f.

My Notes

ACADEMIC VOCABULARY

A **rational function** is a function that is the quotient of two polynomials. Its parent function is $f(x) = \frac{1}{x}$.

TECHNOLOGY TIP

Throughout this book, graphing calculator viewing window dimensions are given as [xmin, xmax] by [ymin, ymax].

ACADEMIC VOCABULARY

A horizontal asymptote is the line $y = a$ when the end behavior of a function approaches some constant a.

A vertical asymptote is the line $x = b$ if the absolute value of a function increases without bound as x approaches some number b.

ACTIVITY 5.4 *Continued*

TEACHER TO TEACHER The last two pages of this activity provide much of the vocabulary and mathematics of rational functions. The context is set aside so students can focus on the math.

First Paragraph Vocabulary Organizer, Interactive Word Wall

18 Create Representations, Note Taking, Debriefing If students have trouble graphing the function, check to see that they enclosed the numerator and denominator in parentheses. Otherwise the calculator will read the function as $f(x) = 12000 + \frac{45x}{x} - 30$. You may also need to review some of the vocabulary with students. If they are stuck, remind them that the process for finding the intercepts of a function has not changed. The y-intercept is $f(0)$ and to find the x-intercept, set the function equal to 0 and solve for x. Students may or may not recall that a function is not defined when you divide by zero, so you might need to work through that part of the Item with them directly if they are unable to proceed.

Last Paragraph Vocabulary Organizer, Interactive Word Wall

MINI-LESSON: Identifying Rational Functions

Struggling students may need additional help to recognize and understanding what is and what is not a rational function. Give them these examples and have them work in groups to sort them into two groups: Rational Function and Not Rational Function.

1. $f(x) = \frac{\sqrt{x} + 2}{3}$ **2.** $f(x) = x^{\frac{5}{4}} + 3x + \frac{1}{2}$ **3.** $f(x) = \frac{2}{x} + 3$

4. $f(x) = x^2 + 3x + \frac{1}{2}$ **5.** $f(x) = \frac{x^2}{3x + 2}$ 3 and 5 are rational functions

19 Think/Pair/Share The horizontal asymptote can be found using the graphing calculator or by performing long division. A later activity will teach students to find horizontal asymptotes by comparing the degree of the numerator and denominator.

20 Look for a Pattern, Create Representations, Group Presentation, Debriefing When debriefing this Item, focus on finding intercepts and asymptotes as key features to identify when sketching a rational function. Later activities will further hone student skills with graphing a rational function without the aid of a calculator. For now, students should use a calculator as needed and make the connections between what they are viewing on the calculator and what they can determine from the equation.

Suggested Assignment

CHECK YOUR UNDERSTANDING
p. 300, #6–8

UNIT 5 PRACTICE
p. 326, #27–30

CHECK YOUR UNDERSTANDING

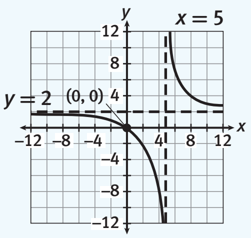
1-6b. See p. 296.

c. See p. 298.

7a. vertical $x = 5$, horizontal $y = 2$

b. x- and y- intercepts are both 0.

c.



8. Answers may vary.

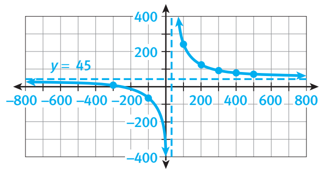
SUGGESTED LEARNING STRATEGIES: Think/Pair/Share, Look for a Pattern, Create Representations, Group Presentation

My Notes

19. What are the vertical and horizontal asymptotes of $f(x) = \dfrac{12000 + 45x}{x - 30}$?

Vertical asymptote: $x = 30$; Horizontal asymptote: $y = 45$.

20. Sketch the graph of the function in Item 19 on the axes below. Indicate the scale, label the intercepts, and include the horizontal and vertical asymptotes.



CHECK YOUR UNDERSTANDING

Write your answers on notebook paper or grid paper. Show your work.

The population of grizzly bears in a remote area is modeled by the function $P(t) = \dfrac{200t - 120}{t + 0.5}$, where $t = 1$ represents the year 2001, $t = 2$ represents the year 2002, and so on.

1. Graph the grizzly bear population function.
2. Describe the features of the graph.
3. What is the domain and range of the function?
4. How many grizzly bears were there in 2005?
5. Predict the bear population in the year 2018.
6. Given the function $f(x) = \dfrac{x + 2}{x - 3}$,
   a. Identify any asymptotes of f.
   b. What are the x- and y-intercepts of f?
   c. Sketch the graph of f.
7. Given the function $f(x) = \dfrac{2x}{x - 5}$,
   a. Identify any asymptotes f.
   b. What are the x- and y-intercepts of f?
   c. Sketch the graph of f.
8. MATHEMATICAL REFLECTION What did you learn about rational functions while working on this activity?

TEACHER TO TEACHER It is possible for a function f to have two horizontal asymptotes: one corresponding to the value approached as x increases with bound; and one as x decreases without bound. The phrase *absolute value of x* in the description on the previous page suggests that only one horizontal asymptote is possible. As an example, the function $f(x) = \dfrac{\sqrt{4x^2 + 5}}{x - 3}$ has two horizontal asymptotes.

Inverse Variation and Rational Functions
Stream Survival

ACTIVITY 5.5

SUGGESTED LEARNING STRATEGIES: Activating Prior Knowledge, Marking the Text, Quickwrite, Create Representations, Look for a Pattern, Group Presentation

My Notes

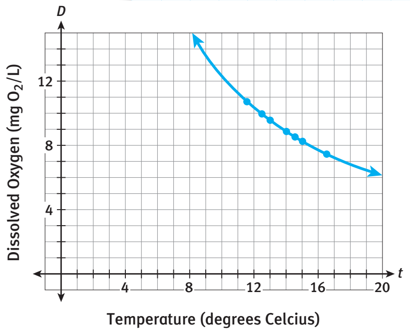
The amount of dissolved oxygen in a body of water decreases as the water temperature increases. Dissolved oxygen needs to be at sufficient levels to sustain the life of aquatic organisms such as fish. The table shows the temperature *t* and the corresponding amount of dissolved oxygen *D* in a stream that flows into Lake Superior on several dates from May to August.

Date	$t°$ (Celsius)	D (mg O_2/L)	Product $t \cdot D$
May 1	11.5	10.6	121.9
May 15	12.5	9.8	122.5
June 1	13.0	9.5	123.5
June 15	14.0	8.7	121.8
July 1	14.5	8.5	123.25
July 15	15.0	8.1	121.5
Aug 1	16.5	7.4	122.1

CONNECT TO ECOLOGY

Fish and other aquatic organisms need oxygen to live, just as mammals do. Dissolved oxygen in the water passes through fish gills and then is transferred into the bloodstream. When dissolved oxygen levels in the water are too low, then not enough oxygen will move into the bloodstream of the fish to maintain life.

1. Graph the data above as a set of points on the axes.
Answer to Item 1 is set of points; curve is answer to Item 7.



2. Are these data linear? Explain why or why not.
Although the data may appear linear from the graph, if you use the table, you can see that slope between pairs of points is not constant.

3. Add a fourth column to the table, showing the product of *t* and *D*.
See above.

4. What do you observe about the products of *t* and *D* that you recorded in the table?
They are very close to each other—between 121.5 and 123.5.

ACTIVITY 5.5 Guided

Inverse Variation and Rational Functions

Activity Focus
- Inverse variation
- Transformations of $f(x) = \frac{1}{x}$
- Characteristics of rational functions
- Modeling of data

Materials
- Graphing calculator

Chunking the Activity

#1–2	#10–11
#3–4	#12–14
#7–9	#15–16
Try These A	#17– Try These C
Example 1–	Example 2–
Try These B	Try These D

Differentiating Instruction

For English learners or students struggling with the context, pre-teach vocabulary related to the context of this problem.

First Paragraph 1 Activating Prior Knowledge, Marking the Text, Create Representations Student graphs may appear linear due to scaling.

2 Quickwrite If groups are not checking the linearity of the data by looking for a constant rate of change, take time to review this concept.

3-4 Create Representations, Look for a Pattern, Quickwrite, Group Presentation Students should observe that all the products are close to 122. If two quantities vary inversely, their product will be a constant. This idea is explained on the next page.

Connect to Ecology

Many factors contribute to water quality, but without enough dissolved oxygen, all aquatic life forms will suffer. In reality, most aquatic life living in a stream or lake would migrate to a cooler or warmer portion of the body of water to meet biological needs. Water temperature varies as the depth increases.

From a theoretical standpoint, the water temperature and dissolved oxygen levels do vary inversely. However, the data in this activity do not exhibit a perfect inverse variation because many factors affect the daily water temperature of a stream over the course of a summer. In general, from early spring through early August, average water temperatures will rise in the geographic locations associated with these data.

Definition Box and First Paragraph Note Taking, Interactive Word Wall, Vocabulary Organizer, Summarize/Paraphrase/Retell Take time to explain the definition of *inverse variation*. Discuss what it means to create a mathematical model. While a model may not fit the data perfectly, it is useful for making predictions and explaining the nature of real-world phenomena.

5 Think/Pair/Share When you debrief this Item, take note of how students decided on their value for k. Some may average the products they found, while others may estimate it in other ways. At this point, accept all methods.

6 Create Representations Some students may have difficulty with using the definition given in terms of x and y if a question uses different variables.

7-8 Create Representations, Quickwrite If necessary, show students how to enter data into their calculators. Have students create two lists, one for the independent values (t) and one for the dependent values (D). Then show students how to use the statistics plotting features to create a scatter plot of the points (t, D). Encourage students to use the graph scale in Item 1 to set up their window. If students select any value for k near 122, their graphs will be very close to the graph for Item 1.

9 Work Backward, Debriefing There are many ways to solve this problem. Some students may try guess and check, others may use algebra or their calculator to graph the line $y = 6$ and find where it intersects their curve. Use whiteboards for the results of Items 7–9 and encourage multiple solution methods.

My Notes

SUGGESTED LEARNING STRATEGIES: Note Taking, Interactive Word Wall, Vocabulary Organizer, Summarize/Paraphrase/Retell, Think/Pair/Share, Create Representations, Quickwrite, Work Backward

ACADEMIC VOCABULARY
inverse variation

Inverse Variation Equation

When the product of two variable quantities x and y is constant, the two variables are said to vary inversely.

If $xy = k$ and $x \neq 0$, then $y = \frac{k}{x}$, where k is the **constant of variation**.

Although the products of t and D from Item 3 are not constant, the products are close in value. When you use mathematics to model a real-world situation, the functions do not always give exact results.

5. If you use inverse variation to model the dissolved oxygen and temperature relationship, what value would you choose for k?
Answers will vary. Sample answer: 122.5, which is the average of 121.5 and 123.5.

6. Write an inverse variation equation relating t and D that shows a constant product. Then solve the equation for D.
$tD = 122.5$
$D = \frac{122.5}{t}$

7. Use your calculator to make a scatter plot of the points (t, D) and graph the equation from Item 6 on the axes in Item 1.
See graph in Item 1 for answer to Item 7.

TECHNOLOGY TIP

Use a graphing calculator to compute a power regression for the data and compare that equation to the one you wrote.

8. How well does the model that you created fit the data?
Answers will vary. Sample answer: It fits the points very well.

9. When dissolved oxygen is less than 6 mg O_2/L, salmon are in danger. Use the model to find the maximum safe temperature for salmon.
Solve: $\frac{122.5}{t} \geq 6$, so $t \leq 20.4$. If the temperatures stay below 20.4 degrees Celsius, then the fish will be safe.

TECHNOLOGY TIP

If students are familiar with using the regression equation features of a calculator, they could easily find a power regression. Their equation should be close to $D = 124t^{-1.04}$, which further confirms that an inverse variation model is appropriate. If students have not used the regression features of the calculator before, you could have them find both a linear and a power model. To better compare the two models, have them graph the two equations and then re-size their viewing window using the ZOOMSTAT. Students will clearly see that a power regression fits the data better than a linear regression when viewing the graph in a smaller window.

Ask these follow-up questions: What is the power of x? Does this confirm that an inverse variation model is appropriate?

SUGGESTED LEARNING STRATEGIES: Create Representations, Marking the Text, Group Presentation, Vocabulary Organizer, Interactive Word Wall, Note Taking

My Notes

TRY THESE A

Use an inverse variation equation to solve each problem.

a. y varies inversely as x. When x is 5, y is 10. Find y when x is 18.
$k = 10 \cdot 5$
$y = \frac{50}{x}$ when $x = 18$, $y = 2.78$

b. The length of a rectangle varies inversely as its width. If the area is 40 in.^{2} and the width is 12.5 in., find the length of the rectangle.
The length is 3.2 inches.

c. Boyle's law says that the volume of a gas in a closed container at constant temperature is inversely proportional to the pressure of the gas. Suppose 5 L of a gas are at a pressure of 2.0 atmospheres. What will be the volume if the pressure is increased to 3.0 atmospheres?
$k = 5 \cdot 2 = 10$
$V = \frac{10}{p}$ when $p = 3$, $V = \frac{10}{3}$ L

Another type of variation is **direct variation**. Two unknowns x and y vary directly if they are related by the equation $y = kx$ where k is a nonzero constant. The graph of a direct variation equation is a line passing through the origin.

MATH TERMS

Other types of variation include **joint variation**, written in the form $y = kxz$, and **combined variation**, written in the form $y = \frac{kx}{z}$. The constant of variation is k in both types of variation.

EXAMPLE 1

The area of a rectangle with a fixed width varies directly as its length. When the area is 40 cm^2, the length is 5 cm. Write a direct variation equation for the area of the rectangle. Use the equation to determine the area when the length is 20 cm.

Step 1: *Use the direct variation formula with A as area, l as length, k as constant of proportionality.* $A = kl$

Step 2: *Substitute A = 40 and l = 5 to find k.* $40 = 5k$; $k = 8$

Step 3: *Write the direct variation equation for this situation.* $A = 8l$

Step 4: *Find A when l = 20.* $A = 8(20)$; $A = 160$

Solution: The area is 160 cm^2.

TRY THESE B

Use a direct variation equation to solve the problem.

a. y varies directly as x. When $x = 3$, $y = 30$. Find y when $x = 7$.
$y = 10x$. When $x = 7$, $y = 70$.

b. Distance traveled varies directly as time if the speed is constant. A 500-mi trip takes 8 h at a constant speed. How long would it take to travel 400 mi at the same speed? $d = 62.5t$. When $d = 400$ miles, $t = 6.4$ hours.

ACTIVITY 5.5 *Continued*

TRY THESE A Create Representations, Marking the Text, Group Presentation, Debriefing These problems provide practice with finding inverse variation equations. If students are having difficulty with these in their groups, use the first one as a whole class example.

Paragraph Vocabulary Organizer, Interactive Word Wall Direct variation is introduced briefly in this activity. For students who are not familiar with direct variation from their Algebra 1 course, review this topic now. See also *Mini-Lesson: Types of Variations* below.

EXAMPLE 1 Note Taking

TRY THESE B Create Representations, Marking the Text, Debriefing

Suggested Assignment

CHECK YOUR UNDERSTANDING
p. 308, #1–7

UNIT 5 PRACTICE
p. 326, #31–34

MINI-LESSON: Types of Variations

Variation problems become easier once students understand the vocabulary.

Words	Basic Equation
y varies directly as x	$y = kx$
y varies jointly as x and z	$y = kxz$
y varies inversely as x	$y = \frac{k}{x}$

Combined variation includes both direct and inverse. For example, V varies directly as x and inversely as the square of y in the equation $V = k\frac{x}{y^2}$.

TEACHER TO TEACHER The focus of the rest of the activity is developing the parent function $f(x) = \frac{1}{x}$ and then using transformations to graph other rational functions. Remind students of their previous work with rational functions, including vocabulary, by reviewing the graph they sketched for Item 20 in Activity 5.4 and pointing out the terms that you have added to your interactive word wall.

10 Vocabulary Organizer, Interactive Word Wall, Create Representations Emphasize the need to graph key points of the function. Students should graph two points at a minimum, (1, 1) and (−1, −1). Until students become familiar with the graph, encourage at least 4 more points in their table and on their graph. They can also use the symmetry of the function to help them sketch the parent function quickly and accurately.

11 Quickwrite, Group Presentation, Debriefing Check how students are using the vocabulary introduced in earlier units. They should be able to describe this function and identify where it is positive and negative. Although students have already studied end behavior, rational functions have different end behavior than polynomials. Make sure students include the asymptotes in their descriptions as well.

Paragraph Vocabulary Organizer, Interactive Word Wall

12 Create Representations Students explore the effect of multiplication by a constant in this Item and the next two Items. Here students should first graph the functions on their graphing calculators to make the graphing easier. Then they should make a quick sketch on the axes provided.

SUGGESTED LEARNING STRATEGIES: Vocabulary Organizer, Interactive Word Wall, Create Representations, Quickwrite, Group Presentation

My Notes

MATH TIP

The basic rational function is sometimes called the *reciprocal function*. It can be easily graphed by plotting the ordered pairs $\left(n, \frac{1}{n}\right)$.

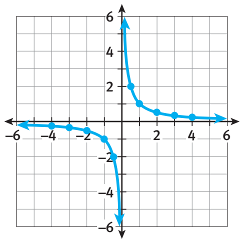
x	$f(x)$
−4	$-\frac{1}{4}$
−3	$-\frac{1}{3}$
−2	$-\frac{1}{2}$
−1	−1
$-\frac{1}{2}$	−2
$\frac{1}{2}$	2
1	1
2	$\frac{1}{2}$
3	$\frac{1}{3}$
4	$\frac{1}{4}$

MATH TIP

Given $y = f(x)$, the function $y = c \cdot f(x)$ represents a *vertical stretch or shrink* of the original function whose y-values have been multiplied by $c > 0$. You have a vertical stretch for $c > 1$ and a vertical shrink for $0 < c < 1$.

The rational function $f(x) = \frac{1}{x}$ is an example of an inverse variation equation whose constant of variation is 1.

10. Make a table of values in the My Notes section. Graph the parent rational function $f(x) = \frac{1}{x}$ below.

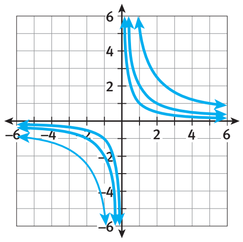


11. Describe the key features of $f(x) = \frac{1}{x}$. Use appropriate mathematics vocabulary in your description.

Answers will vary. Sample answer: The function has a vertical asymptote at $x = 0$ and a horizontal asymptote at $y = 0$. The domain is all real numbers except for $x = 0$, and the range is all real numbers except for $y = 0$. When $x < 0$, the function is negative. When $x > 0$, the function is positive.

Functions like the one modeling dissolved oxygen and temperature are a *vertical stretch* of the parent graph $f(x) = \frac{1}{x}$.

12. Enter the functions $f(x) = \frac{1}{x}$, $g(x) = \frac{2}{x}$, and $h(x) = \frac{5}{x}$ into your graphing calculator. Sketch the graphs on the axes below.



TEACHER TO TEACHER There is a relationship between the natural logarithms and the rational function $f(x) = \frac{1}{x}$. The area under the function f from 1 to any number $a > 0$ is equal to the ln a. Students will actually define the natural logarithm function using a definite integral in calculus.

SUGGESTED LEARNING STRATEGIES: Look For a Pattern, Quickwrite, Create Representations, Predict and Confirm, Vocabulary Organizer, Interactive Word Wall

My Notes

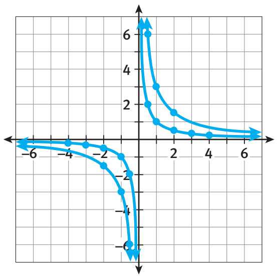
13. How do the y-values of g and h compare to those of the parent graph?

Answers will vary. Sample answer: The y-values of g are 2 times the parent function's y-values. The y-values of h are 5 times the parent function's y-values.

14. Describe the similarities and the differences in the graphs of those three functions.

All three graphs have the same vertical and horizontal asymptotes. As the coefficient gets larger, the graph is stretched farther away from the axes.

15. Sketch the parent graph $f(x) = \frac{1}{x}$ and the graph of $k(x) = \frac{3}{x}$ on the same axes without using your graphing calculator.



16. Without using your calculator, predict what the graph of $f(x) = -\frac{1}{x}$ will look like. Confirm prediction by graphing both functions on your calculator.

Answers will vary. Sample answer: The graph will be reflected across the x-axis.

MATH TIP

Given $y = f(x)$, the function $y = -f(x)$ represents a *vertical reflection* of the original function whose y-values have been multiplied by -1.

ACTIVITY 5.5 *Continued*

13-14 **Look for a Pattern, Quickwrite, Debriefing**
Students explore the graphs they made in Item 12. While they may describe the graphs as moving away from the origin, make sure when you debrief that students understand that a vertical stretch has taken place. Some students may say that the graphs appear steeper than the parent function near $x = 0$ or not as steep as the parent function as x increases.

15 **Create Representations**
This Item provides a formative assessment opportunity. Check to see that students are using key points to sketch the parent function and then multiplying the y-values of those points by 3 to get the y-values on the new function.

16 **Predict and Confirm, Quickwrite, Debriefing**
The fact that $-f(x)$ is a reflection of $f(x)$ across the x-axis should be very familiar to students at this point. They are only confirming that it still applies to rational functions.

Vocabulary Feature Vocabulary Organizer, Interactive Word Wall

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17 Create Representations, Look for a Pattern, Note Taking Students practice graphing and have an opportunity to explore translations of rational functions in this Item. When groups report out their work on whiteboards or chart paper, make sure to reinforce and recognize those groups who used proper vocabulary, those who translated the asymptotes and labeled them, and those who used their key points to create their graphs.

Vocabulary Feature Vocabulary Organizer, Interactive Word Wall

SUGGESTED LEARNING STRATEGIES: Create Representations, Look For a Pattern, Note Taking, Vocabulary Organizer, Interactive Word Wall

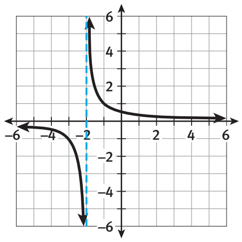
My Notes

MATH TERMS

Given $y = f(x)$, the function $y = f(x + c)$ results in a *horizontal translation* of the original function and $y = f(x) + c$ results in a *vertical translation* of the original function

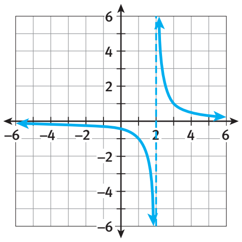
17. Sketch the graph of each function and then describe it as a transformation of the parent graph $f(x) = \frac{1}{x}$. The first graph has been done for you.

$f(x) = \frac{1}{x+2}$



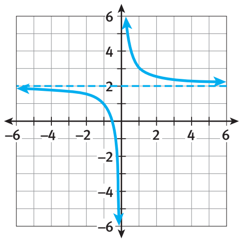
Transformation: The graph is translated 2 units to the left.

$f(x) = \frac{1}{x-2}$



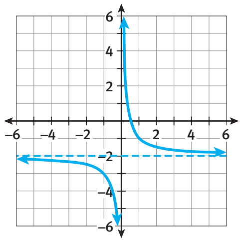
Transformation: The graph is translated 2 units to the right.

$f(x) = \frac{1}{x} + 2$



Transformation: The graph is translated up 2 units.

$f(x) = \frac{1}{x} - 2$



Transformation: The graph is translated down 2 units.

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306 SpringBoard® Mathematics with Meaning™ Algebra 2

SUGGESTED LEARNING STRATEGIES: Create Representations, Look for a Pattern, Quickwrite, Note Taking, Identify a Subtask, Graphic Organizer

My Notes

TRY THESE C

Describe each function as a transformation of $f(x) = \frac{1}{x}$.

a. $f(x) = \frac{1}{x+1}$ The graph is translated 1 unit to the left.

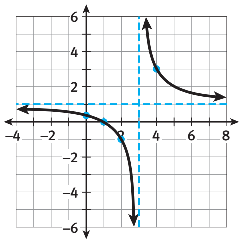
b. $f(x) = \frac{1}{x} - 3$ The graph is translated down 3 units.

c. $f(x) = \frac{1}{x-5} + 3$ The graph is translated 5 units to the right and up 3 units.

EXAMPLE 2

Describe the function $f(x) = \frac{2}{x-3} + 1$ as a transformation of $f(x) = \frac{1}{x}$.

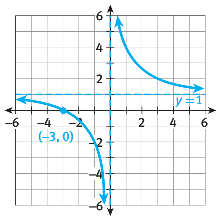
Identify the x- and y-intercepts and the asymptotes. Sketch the graph.

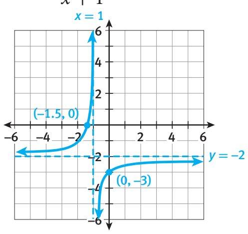
Transformations	**Asymptotes**
• vertical stretch by a factor of 2 • horizontal translation 3 units to the right • vertical translation 1 unit up	$x = 3$ $y = 1$
Intercepts y-intercept: $f(0) = \frac{1}{3}$ x-intercept: Solve $f(x) = 0$. $\frac{2}{x-3} + 1 = 0$ $\frac{2}{x-3} = -1$ $2 = -1(x-3)$ $x = 1$	**Graph** 

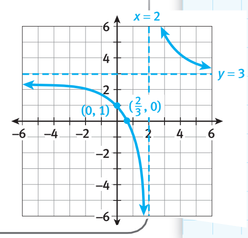
TRY THESE D

On a separate sheet of grid paper, describe each function as a transformation of $f(x) = \frac{1}{x}$. Identify the x- and y-intercepts and the asymptotes. Sketch the graph.

a. $f(x) = \frac{3}{x} + 1$ **b.** $f(x) = -\frac{1}{x+1} - 2$ **c.** $f(x) = 3 + \frac{4}{x-2}$







b. Reflected about the x-axis, translated 1 unit left and 2 down.
Vertical Asymptote: $x = -1$,
Horizontal Asymptote: $y = -2$,
x-intercept -1.5,
y-intercept -3

c. Vertically stretched by a factor of 4, translated 2 units right and 3 up.
Vertical Asymptote: $x = 2$,
Horizontal Asymptote: $y = 3$,
x-intercept $\frac{2}{3}$, y-intercept 1.

ACTIVITY 5.5 *Continued*

TRY THESE C Create Representations, Look for a Pattern, Quickwrite Use these Items to determine whether or not students can identify and describe the translations correctly. If time permits, have students graph these in their notebooks. Be sure that students are translating the asymptotes and using key points to make their graphs. It is easier if students draw the asymptotes first and then sketch the function around them.

EXAMPLE 2 Note Taking, Identify a Subtask, Graphic Organizer, Create Representations Students put all the transformations together, find the intercepts, and graph the functions as well. Work through the example with the class before students begin the Try These problems on their own.

TRY THESE D Identify a Subtask, Quickwrite, Create Representations In Part (c), the constant appears before the function. This is a good opportunity to remind students of the Commutative Property of Addition.

Differentiating Instruction

If necessary, guide students through the algebra of finding the x-intercepts on one or more problems. A graphic organizer like the one shown in the example will also help students break the graphing into manageable chunks.

Suggested Assignment

CHECK YOUR UNDERSTANDING
p. 308, #8–13

UNIT 5 PRACTICE
pp. 326–327, #35–39

TRY THESE D

a. Vertically stretched by a factor of 3 and translated up 1 unit.
Vertical Asymptote: $x = 0$,
Horizontal Asymptote:
$y = 1$, x-intercept -3,
no y-intercept.

CHECK YOUR UNDERSTANDING

1. 20
2. Let t = time and r = rate
   $1.5 \cdot 50 = 75$
   $t = \frac{75}{r}$
3. The distance is 75 miles.
4. The trip will take about 1.15 h.
5. Yes, it does, because the products of x and y are constant, 48.
6. No, it does not. The data have a constant slope, so they are linear, but quotients of the ordered pairs are not constant, so it doesn't fit the form $y = kx$.
7. $y = 0.25x$. When $x = 5$, $y = 1.25$.
8. The graph is stretched vertically by a factor of 2 and translated 5 units to the left.
9. The graph is reflected across the x-axis and translated 3 units to the right and up 5 units.
10. The graph is stretched by a factor of 10 and translated 4 units to the left and down 2 units.
11. and 12. See below.
13. Answers may vary.

CHECK YOUR UNDERSTANDING

Write your answers on notebook paper or grid paper. Show your work.

1. y varies inversely as x. When $y = 5$, $x = 20$. Find y when $x = 5$.

In Items 2–4, the time to travel a fixed distance varies inversely as speed.

2. Write an inverse variation model for a trip that takes 1.5 h when you average 50 mph.
3. Interpret the meaning of the constant of variation.
4. Use the equation to determine how long the trip takes if you average 65 mph.
5. Does the data below represent an inverse variation model? Explain why or why not.

x	2	4	6	8
y	24	12	8	6

6. Does the data below represent a direct variation model? Explain why or why not.

x	2	4	6	8
y	6	10	14	18

7. y varies directly as x. When $y = 5$, $x = 20$. Find y when $x = 5$.

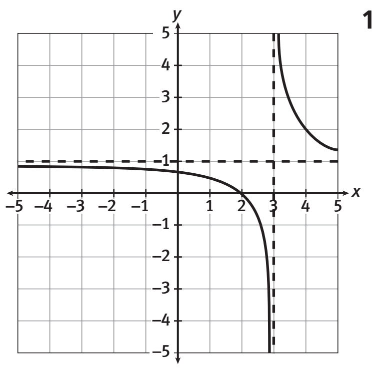
Describe each function as a transformation of $f(x) = \frac{1}{x}$.

8. $f(x) = \frac{2}{x+5}$
9. $f(x) = -\frac{1}{x-3} + 5$
10. $f(x) = \frac{10}{x+4} - 2$

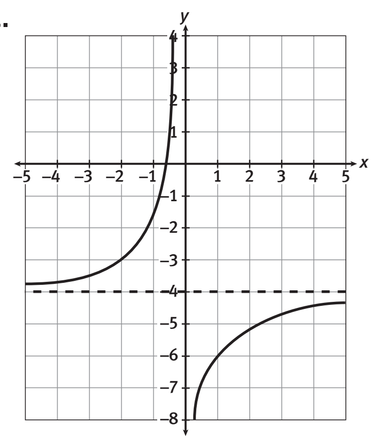
Graph each rational function, using your knowledge of transformations.

11. $f(x) = \frac{1}{x-3} + 1$
12. $f(x) = -\frac{2}{x} - 4$
13. MATHEMATICAL REFLECTION How can you use transformations to graph rational functions?

11.



12.



Simplifying Rational Expressions
It's All Rational

ACTIVITY 5.6

SUGGESTED LEARNING STRATEGIES: Summarize/Paraphrase/Retell, Interactive Word Wall, Vocabulary Organizer, Note Taking, Think/Pair/Share, Create Representations, Simplify the Problem

My Notes

Rational expressions can be simplified and combined, using the operations of addition, subtraction, multiplication and division.

Writing rational expressions in simpler forms and combining them helps you to understand and graph rational functions and solve equations.

To simplify a rational expression, factor the numerator and denominator. Identify the restrictions on the variable x that make the denominator in the expression equal to zero. Then, divide out the common factors.

MATH TIP

When a rational function has a denominator of zero, restrictions on the variable are needed.

EXAMPLE 1

Simplify each expression.

		A.	B.
	Original expression	$\frac{x^2+5x-14}{x^2-4}$	$\frac{2x^2+7x+3}{x^2+7x+12}$
Step 1:	*Identify the restrictions on x. Set the denominators equal to zero.*	$x^2-4=0$ $(x+2)(x-2)=0$ $x+2=0$ or $x-2=0$ $x=-2$ or $x=2$	$x^2+7x+12=0$ $(x+3)(x+4)=0$ $x+3=0$ or $x+4=0$ $x=-3$ or $x=-4$
Step 2:	*Factor the numerators and denominators.*	$\frac{x^2+5x-14}{x^2-4}=$ $\frac{(x+7)(x-2)}{(x+2)(x-2)}=$	$\frac{2x^2+7x+3}{x^2+7x+12}=$ $\frac{(2x+1)(x+3)}{(x+4)(x+3)}=$
Step 3:	*Divide out common factors.*	$\frac{(x+7)\cancel{(x-2)}}{(x+2)\cancel{(x-2)}}=$ $\frac{x+7}{x+2}, x\neq 2, -2$	$\frac{(2x+1)\cancel{(x+3)}}{(x+4)\cancel{(x+3)}}=$ $\frac{2x+1}{x+4}, x\neq -3, -4$

TRY THESE A

Simplify. Identify any restrictions on x. Write your answers on notebook paper. Show your work.

a. $\frac{x^2+20x+36}{x^3-4x}$ $\quad \frac{x+18}{x(x-2)}, x\neq 0, -2, 2$

b. $\frac{x^2-2x-15}{2x^2+3x-9}$ $\quad \frac{x-5}{2x-3}, x\neq \frac{3}{2}, -3$

c. $\frac{x^3-9x}{3-x}$ $\quad -x(x+3), x\neq 3$

MATH TIP

Make sure that you use the original rational expression when identifying restrictions on the variable.

TEACHER TO TEACHER To understand the restrictions on the variables in Examples 1–3, students should think of these rational expressions as functions. As functions, each has a domain. Students then determine the domain of each rational function (i.e., the restrictions on the variable represent values that are not part of the domain). Once the domain has been identified, common factors can be cancelled, provided that the new expression still preserves the domain that it previously had. In the context of functions, all this makes more sense, because two functions are equal if their domains are identical and their values are equal over their identical domains.

ACTIVITY 5.6 Directed

Simplifying Rational Expressions

Activity Focus

- Simplifying rational expressions
- Operations on rational expressions
- Graphing rational functions

Materials

- Graphing calculator

Chunking the Activity

Example 1–Try These A
Example 2–Try These B
Example 4–Try These D
Example 6–Try These E
Example 7–Try These G
Example 9
Example 10–Try These H

TEACHER TO TEACHER This activity assumes proficiency with factoring expressions. If necessary, review factoring and provide additional practice.

First Three Paragraphs Summarize/Paraphrase/Retell, Interactive Word Wall, Vocabulary Organizer In this activity, the phrases *cancel* and *divide out* refer to the elimination of common factors in the numerator and denominator. Be sure to clarify that this process uses a quotient of common factors, which is equal to 1, the multiplicative identity.

EXAMPLE 1 Note Taking Model the example and check for understanding. Remind students to identify restricted values before they cancel any common factors.

TRY THESE A Think/Pair/Share, Create Representations, Simplify the Problem, Debriefing

TEACHER TO TEACHER When multiplying two rational expressions, some students fail to cancel common factors that appear in the numerator and denominator of the same expression. They may be confusing cancelling a common factor with cross-multiplying in proportions.

The multiplicative identity does not apply to the operation of addition. For example, $\frac{x+2}{x+3} \neq \frac{2}{3}$. Students must understand that they can cancel only factors in a product, not addends in a sum.

First Paragraph Summarize/ Paraphrase/Retell

EXAMPLES 2–3 Note Taking
Identification of restricted values was specifically excluded from these problems because many restricted values occur, especially when dividing rational expressions. The denominator of the dividend cannot be zero and neither the numerator nor the denominator of the divisor can be zero.

Second Paragraph Marking the Text

TRY THESE **B** Think/Pair/Share, Create Representations, Group Presentation

TEACHER TO TEACHER It is important to have students identify all the restricted values throughout this activity. In Example 3 on this page, the variable cannot equal the following values: −3, −2, and 2.

Suggested Assignment

CHECK YOUR UNDERSTANDING
p. 316, #1–5

UNIT 5 PRACTICE
p. 327, #40–42

Simplifying Rational Expressions
It's All Rational

SUGGESTED LEARNING STRATEGIES: Summarize/ Paraphrase/Retell, Note Taking, Marking the Text, Think/ Pair/Share, Create Representations, Group Presentation

My Notes

To multiply rational expressions and express the product in lowest terms, factor the numerator and denominator of each expression. Then, divide out any common factors.

EXAMPLE 2

Multiply the expression. Assume no denominator is zero.

Original expression $\frac{2x^2-8}{x^2-1} \cdot \frac{x^2+2x+1}{x^3-x^2-2x}$

Step 1: *Factor the numerators and denominators.* $\frac{2(x+2)(x-2)}{(x+1)(x-1)} \cdot \frac{(x+1)(x+1)}{x(x-2)(x+1)}$

Step 2: *Divide out common factors.* $\frac{2(x+2)\cancel{(x-2)}\cancel{(x+1)}\cancel{(x+1)}}{\cancel{(x+1)}(x-1)x\cancel{(x-2)}\cancel{(x+1)}}$

$\frac{2(x+2)}{x(x-1)}$

MATH TIP

When dividing numerical fractions, write as multiplication.

$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$

If a, b, c, and d have any common factors, you can cancel them before you multiply.

$\frac{4}{15} \div \frac{8}{3} = \frac{4}{15} \cdot \frac{3}{8} =$
$\frac{\cancel{4}}{\cancel{3} \cdot 5} \cdot \frac{\cancel{3}}{2 \cdot \cancel{4}} = \frac{1}{10}$

To divide rational expressions, write division as multiplication and then finish simplifying the expression.

EXAMPLE 3

Divide the expression. Assume no denominator is zero.

Original expression $\frac{x^2+5x+6}{x^2-4} \div \frac{5x+15}{3x^2-4x-4}$

Step 1: *Write as multiplication.* $\frac{x^2+5x+6}{x^2-4} \cdot \frac{3x^2-4x-4}{5x+15}$

Step 2: *Factor the numerators and denominators.* $\frac{(x+2)(x+3)}{(x+2)(x-2)} \cdot \frac{(3x+2)(x-2)}{5(x+3)}$

Step 3: *Divide out common factors.* $\frac{\cancel{(x+2)}\cancel{(x+3)}(3x+2)\cancel{(x-2)}}{\cancel{(x+2)}\cancel{(x-2)}(5)\cancel{(x+3)}}$

$\frac{3x+2}{5}$

TRY THESE B

Perform the indicated operation. Assume no denominator is zero. Write your answers on notebook paper. Show your work.

a. $\frac{2x+4}{x^2-25} \cdot \frac{x^2-5x-50}{4x^2-16}$ $\frac{x-10}{2(x-5)(x-2)}$

b. $\frac{6x^2}{3x^2-27} \div \frac{2x+2}{x^2-2x-3}$ $\frac{x^2}{x+3}$

MINI-LESSON: Opposites

Problem (c) in Try These A on the previous page required students to recognize that the expression $3 - x$ can be written as $-1(x - 3)$. The expressions $3 - x$ and $x - 3$ are additive inverses. When simplifying, the quotient of an expression and its additive inverse will always equal −1. Each step in the solution of Problem (c) is shown below.

$$\frac{x^3-9x}{3-x} = \frac{x(x-3)(x+3)}{-(-3+x)} = \frac{x\cancel{(x-3)}(x+3)}{-\cancel{(x-3)}} = \frac{x(x+3)}{-1} = -x(x+3)$$

SUGGESTED LEARNING STRATEGIES: Summarize/Paraphrase/Retell, Note Taking, Graphic Organizer, Think/Pair/Share

My Notes

To add or subtract rational expressions with unlike denominators, find a common denominator. The easiest way to find the least common denominator is to factor the expressions. Then, the least common denominator is the product of each factor common to the expressions and any non-common factors.

EXAMPLE 4

Find the least common denominator of $\frac{1}{x^2 - 3x - 4}$ and $\frac{1}{x^2 - 16}$.

Step 1: *Factor each denominator.* $x^2 - 3x - 4 = (x + 1)(x - 4)$
$x^2 - 16 = (x + 4)(x - 4)$

Step 2: *Identify common factors and factors not in common.* Factors in Common: $x - 4$
Factors Not in Common: $x + 4, x + 1$

Step 3: *Write the least common denominator.* $(x + 4)(x + 1)(x - 4)$

TRY THESE C

Find the least common denominator of $\frac{1}{x^2 - 9}$ and $\frac{1}{3x^2 - 9x}$.

$3x(x - 3)(x + 3)$

Now you are ready to add and subtract rational expressions with different denominators.

EXAMPLE 5

Simplify the expression. Assume no denominator is zero.

Original expression $\frac{2}{x - 2} - \frac{3}{x^2 - 2x} =$

Step 1: *Factor the denominators.* $\frac{2}{x - 2} - \frac{3}{x(x - 2)}$

Step 2: *Find the least common denominator.* $x(x - 2)$

Step 3: *Multiply numerator and denominator of each term by the missing factor(s) of the least common denominator.* $\frac{2(x)}{x(x - 2)} - \frac{3}{x(x - 2)} =$

Step 4: *Subtract the like fractions to find the solution.* $\frac{2x - 3}{x(x - 2)}$

MATH TIP

When the denominators are the same, all you have to do is add or subtract the numerators as indicated by the operation.

CONNECT TO AP

You will continue to use the skill of simplifying rational expressions in AP Calculus.

ACTIVITY 5.6 *Continued*

TEACHER TO TEACHER No problems have been provided where the denominators are the same. A few extra exercises are provided in *Mini-Lesson* below.

First Paragraph Summarize/Paraphrase/Retell

EXAMPLE 4 Note Taking, Graphic Organizer Students having difficulty may find that using a graphic organizer helps to identify the simplest common denominator. When the denominators have no common factors, the common denominator will be the product of all the denominators.

TRY THESE C Think/Pair/Share, Debriefing

EXAMPLE 5 Note Taking Ask students why the second rational expression is unchanged after the common denominator is found. Point out that this is only the case when the denominator of a rational expression is equal to the simplest common denominator.

MINI-LESSON: Adding and Subtracting with Like Denominators

Use these problems if students need additional practice with adding and subtracting rational expressions with like denominators.

1. $\frac{3 + 2x}{x + 1} + \frac{3x - 1}{x + 1}$ $\frac{5x + 2}{x + 1}$
2. $\frac{3}{x - 4} - \frac{3 + 4x}{x - 4}$ $-\frac{4x}{x - 4}$
3. $\frac{x^2}{x + 1} - \frac{1}{x + 1}$ $x - 1$

TRY THESE D Think/Pair/Share, Simplify the Problem, Group Presentation Have students present their work to the entire class. You can circulate from group to group as they work, asking and answering questions that will assist students having difficulty.

TEACHER TO TEACHER Again, the restrictions on the variable are omitted in these problems because of the difficulty in finding them. Remind students that the restrictions are taken from the original problem, not the simplified expression. In Example 6, the restrictions are $x \neq -1, 0, 1, 2$. Notice that the simplified expression does not have the restriction $x \neq 1$. However, the original expression is not equal to the simplified expression when $x = 1$, because the original expression is undefined when $x = 1$.

First Paragraph Summarize/ Paraphrase/Retell, Vocabulary Organizer, Interactive Word Wall

EXAMPLE 6 Note Taking The strategy presented to simplify complex fractions involves simplifying the numerator and denominator separately until they form a single rational expression.

Another strategy is to multiply by an appropriate form of the number 1. Select all factors to be eliminated from the denominators of the numerator and denominator of the complex fraction. In this case, you would multiply by $\frac{(x+1)(x-1)}{(x+1)(x-1)}$ as shown in the *Mini-Lesson*.

TRY THESE E Create Representations, Simplify the Problem, Group Presentation

My Notes

SUGGESTED LEARNING STRATEGIES: Think/Pair/Share, Simplify the Problem, Group Presentation, Summarize/ Paraphrase/Retell, Vocabulary Organizer, Interactive Word Wall, Note Taking, Create Representations

TRY THESE D

Simplify each expression. Assume no denominator is zero. Write your answers on notebook paper. Show your work.

a. $\frac{3}{x+1} - \frac{x}{x-1}$ $\quad \frac{-x^2+2x-3}{(x+1)(x-1)}$

b. $\frac{2}{x} - \frac{3}{x^2-3x}$ $\quad \frac{2x-9}{x^2-3x}$

c. $\frac{2}{x^2-4} + \frac{x}{x^2+4x+4}$ $\quad \frac{x^2+4}{(x+2)^2(x-2)}$

ACADEMIC VOCABULARY

A rational expression that contains rational expressions in its numerator and/or its denominator is called a **complex fraction.**

You can simplify **complex fractions** if you treat them like a division problem. Simplify the numerator and denominator as much as possible, and then write the problem using multiplication.

EXAMPLE 6

Simplify $\frac{1+\frac{1}{x+1}}{x-\frac{x}{x-1}}$. Assume no denominator is zero.

Original expression $\quad \frac{1+\frac{1}{x+1}}{x-\frac{x}{x-1}} =$

Step 1: *Simplify the numerator and denominator using their least common denominators.* $\quad \frac{\frac{x+1}{x+1}+\frac{1}{x+1}}{\frac{x(x-1)}{(x-1)}-\frac{x}{x-1}} =$

Step 2: *Add or subtract fractions in the numerator and denominator. Combine like terms.* $\quad \frac{\frac{x+1+1}{x+1}}{\frac{x^2-x-x}{x-1}} = \frac{\frac{x+2}{x+1}}{\frac{x^2-2x}{x-1}} =$

Step 3: *Write division as multiplication of the reciprocal.* $\quad \frac{x+2}{x+1} \cdot \frac{x-1}{x^2-2x} =$

Step 4: *Factor and simplify if possible.* $\quad \frac{(x+2)(x-1)}{x(x+1)(x-2)}$

TRY THESE E

Simplify. Assume no denominator is zero. Write your answers on notebook paper. Show your work.

a. $\frac{\frac{x^2-3x-4}{x^2-4}}{\frac{2x^2+2x}{x+2}}$ $\quad \frac{(x-4)}{2x(x-2)}$

b. $\frac{\frac{x}{x+1}-\frac{1}{x-1}}{\frac{1}{x+1}+2}$ $\quad \frac{x^2-2x-1}{(x-1)(2x+3)}$

Suggested Assignment

CHECK YOUR UNDERSTANDING
p. 316, #6–9

UNIT 5 PRACTICE
p. 327, #43–44

MINI-LESSON: Alternate Method for Simplifying Complex Fractions

In Example 6, students can first multiply by 1 using factors that will cancel all denominators in the original complex fraction.

$$\frac{1+\frac{1}{x+1}}{x-\frac{x}{x-1}} = \frac{1+\frac{1}{x+1}}{x-\frac{x}{x-1}} \cdot \frac{(x+1)(x-1)}{(x+1)(x-1)} = \frac{(x+1)(x-1)+(x-1)}{x(x+1)(x-1)-x(x+1)} = \frac{(x-1)(x+2)}{x(x+1)(x-2)}$$

SUGGESTED LEARNING STRATEGIES: Vocabulary Organizer, Note Taking, Think/Pair/Share, Look for a Pattern, Marking the Text, Shared Reading

My Notes

In the graph of a rational function, a break in the graph often signals that a **discontinuity** has occurred. Algebraically, a discontinuity happens for values of x that cause the function to be undefined and are therefore not in the domain of the function.

EXAMPLE 7

Identify any vertical asymptotes in the graph.

Step 1: *Factor the numerator and denominator.*

$$f(x) = \frac{x^2 - 4}{x^2 + 5x + 6}$$

$$f(x) = \frac{(x+2)(x-2)}{(x+2)(x+3)}$$

Step 2: *Divide out the common factors.*

$$f(x) = \frac{x-2}{x+3}$$

Step 3: *Find the values that make the simplified denominator = 0.*

$x + 3 = 0$ when $x = -3$

vertical asymptote $x = -3$

MATH TIP

To find the vertical asymptote of a graph, determine the values of the variables that make the function undefined when it is in simplest form.

TRY THESE F

Identify any vertical asymptotes in the graph.

a. $f(x) = \dfrac{x^2 - x}{x^2 + 3x - 4}$

hole at $x = 1$,
Vertical Asymptote at $x = -4$

b. $f(x) = \dfrac{3 - x}{9 - x^2}$

hole at $x = 3$,
Vertical Asymptote at $x = -3$

A *horizontal asymptote* depends on the degrees of the numerator and denominator and describes the end behavior of a rational function.

- When the degrees are the same, the horizontal asymptote is the ratio of the leading coefficients.
- When the denominator degree is larger, the horizontal asymptote is equal to 0.
- When the numerator degree is larger, there is no horizontal asymptote.

TECHNOLOGY TIP

Use a graphing calculator to graph the function and visually see the breaks where the asymptotes are located.

EXAMPLE 8

Identify the horizontal asymptote, if any.

a. $f(x) = \dfrac{2 + x}{x^2 - 1}$

numerator degree = 1
denominator degree = 2

$2 > 1$

horizontal asymptote: $y = 0$

b. $f(x) = \dfrac{2x + 2}{x - 1}$

numerator degree = 1
denominator degree = 1

lead coefficients: 2, 1
ratio of lead coefficients: 2

horizontal asymptote: $y = 2$

ACTIVITY 5.6 *Continued*

TEACHER TO TEACHER This page provides students with practice identifying vertical and horizontal asymptotes before they are asked to sketch rational functions. Slant asymptotes will be introduced in precalculus.

First Paragraph Vocabulary Organizer, Interactive Word Wall

EXAMPLE 7 Note Taking
Students should know how to find the vertical asymptote from earlier work. The formal statement for finding vertical asymptotes of a rational function is:
If $f(x) = \frac{p(x)}{q(x)}$, where p and q are polynomial functions in standard form with no common factors other than 1, then the function f has a vertical asymptote at each value for x for which $q(x) = 0$.

TRY THESE F Think/Pair/Share, Look for a Pattern, Debriefing

Second Paragraph Vocabulary Organizer, Interactive Word Wall, Marking the Text, Shared Reading

EXAMPLE 8 Note Taking

TECHNOLOGY TIP

While the examples on this page do not involve graphing, a graphing calculator is a valuable tool to reinforce function end behavior and to understand asymptotes. Have students begin with examining the behavior near 0 ($x < 0$ and $x > 0$) for the functions $\frac{1}{x}$ and $\frac{1}{x^2}$. Students should use their number sense to reflect on what happens to $\frac{1}{x}$ as x gets very close to 0 through positive values and through negative values. The y-axis becomes an example of a vertical asymptote in this case.

TRY THESE G Think/Pair/Share, Look for a Pattern, Group Presentation

TEACHER TO TEACHER By the end of this lesson, students should be adept at identifying the key features of a rational function and then using those features to construct a graph.

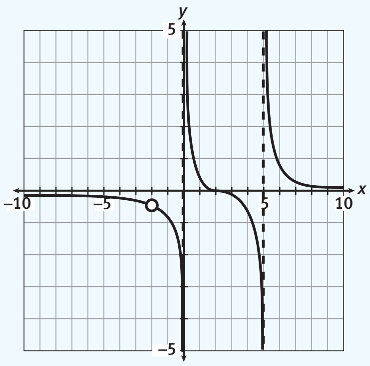
At the Algebra 2 level, encourage students to use a graphing calculator to fill in the missing portions of the graph and confirm their work, after they have sketched in all the key features. After sufficient practice graphing rational functions with a graphing utility, students will recognize patterns and features that will allow them to sketch rational functions without needing to use a graphing utility.

Paragraph and Bulleted List Shared Reading, Questioning the Text Post the steps for graphing rational functions in a prominent location and refer to them often.

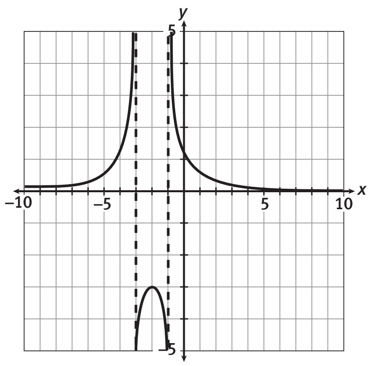
EXAMPLE 9 Graphic Organizer, Note Taking The graphic organizer shown in the example can be used to help students keep track of their work.

TRY THESE H

a.



b.



Simplifying Rational Expressions
It's All Rational

My Notes

SUGGESTED LEARNING STRATEGIES: Think/Pair/Share, Look for a Pattern, Group Presentation, Shared Reading, Questioning the Text, Graphic Organizer, Note Taking

TRY THESE G

Identify the horizontal asymptote, if any.

a. $f(x) = \frac{2-x}{x+4}$ $y = -1$

b. $f(x) = \frac{x^2-1}{x+3}$ none

c. $\frac{x}{x^2-4}$ $y = 0$

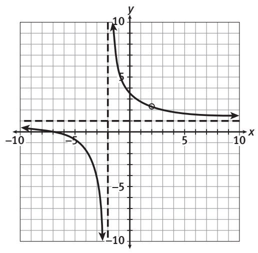
Now you are ready to use your knowledge of simplifying rational expressions to help you understand and graph rational functions.

To graph rational functions, follow these steps.

- Simplify the rational function.
- Express the numerator and denominator in factored form.
- Identify vertical asymptotes.
- Identify x- and y-intercepts.
- Identify horizontal asymptote (end behavior).
- Make a sketch, using a graphing calculator as needed.

EXAMPLE 9

Analyze and graph the rational function $f(x) = \frac{x^2+5x-14}{x^2-4}$.

Simplify. $\frac{x^2+5x-14}{x^2-4} = \frac{(x+7)(x-2)}{(x+2)(x-2)} = \frac{(x+7)\cancel{(x-2)}}{(x+2)\cancel{(x-2)}} = \frac{x+7}{x+2}$	**Identify vertical asymptotes.** $x + 2 = 0$, so vertical asymptote is $x = -2$
	Identify intercepts. x-intercept: $x + 7 = 0$, so $x = -7$; y-intercept: $f(0) = \frac{0+7}{0+2} = 3.5$
Identify horizontal asymptote. numerator degree = 1; denominator degree = 1; lead coefficients: 1, 1; ratio of lead coefficients: 1; horizontal asymptote: $y = 1$	**Graph.** 

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SUGGESTED LEARNING STRATEGIES: Graphic Organizer, Note Taking, Identify a Subtask, Create Representations, Group Presentation

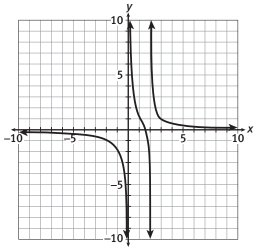
My Notes

MATH TIP

To determine when a sum or difference of rational expressions is 0 or undefined, it helps to combine them into a single expression first.

EXAMPLE 10

Analyze and graph the rational function $f(x) = \frac{2}{x-2} - \frac{3}{x^2-2x}$.

Simplify. $\frac{2}{x-2} - \frac{3}{x^2-2x} =$ $\frac{2}{x-2} - \frac{3}{x(x-2)} =$ $\frac{2(x)}{x(x-2)} - \frac{3}{x(x-2)} =$ $\frac{2x-3}{x(x-2)}$	**Identify vertical asymptotes.** $x = 0$ and $x - 2 = 0$, so vertical asymptotes are $x = 0$ and $x = 2$
	Identify intercepts. x-intercept: $2x - 3 = 0$, so $x = 1.5$ y-intercept: none, because $f(x)$ is undefined when $x = 0$
Identify horizontal asymptote. numerator degree = 1 denominator degree = 2 horizontal asymptote is $y = 0$	**Graph.** 

TRY THESE H

Analyze and graph each rational function. Write your answers on grid paper. Show your work. See side column, page 314 for graphs.

a. $f(x) = \frac{x^2-4}{x^3-3x^2-10x}$

x-intercept (2, 0), no y-intercept, hole at $x = -2$, vertical asymptote at $x = 0$, $x = 5$, horizontal asymptote at $y = 0$

b. $f(x) = \frac{1}{x+1} - \frac{2}{x+3}$

x-intercept (1, 0), y-intercept $\left(0, \frac{1}{3}\right)$, no holes, vertical asymptote at $x = -3$, $x = -1$, horizontal asymptote at $y = 0$.

ACTIVITY 5.6 *Continued*

EXAMPLE 10 Graphic Organizer, Note-Taking Again, a graphic organizer is used to help students with their work. In this example, students must perform an operation on rational expressions before graphing the function.

TRY THESE H Graphic Organizer, Indentify a Subtask, Create Representations, Group Presentation, Debriefing

Differentiating Instruction

Students having difficulty may need to work with simpler examples like the ones below. Since the focus of this activity is graphing rational functions, it may be helpful to substitute these problems if simplifying and then graphing rational functions poses too many difficulties for students.

ALTERNATE EXAMPLES

Analyze and graph each rational function.

1. $f(x) = \frac{x}{x^2-4}$

x-intercept: 0 Vertical Asymptote: $x = 2$ and $x = -2$
Horizontal Asymptote: $y = 0$
Graph shown below.

2. $f(x) = \frac{x+3}{x^2+4x+3}$

no x-intercept, y-intercept: 1
Vertical Asymptote: $x = -1$
Horizontal Asymptote: $y = 0$
Graph shown below.

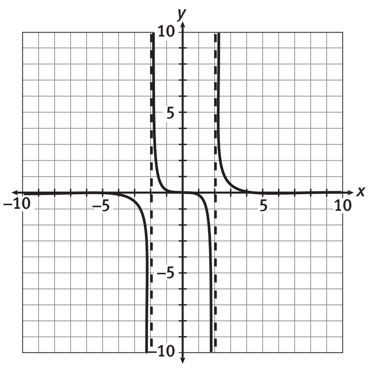
Suggested Assignment

CHECK YOUR UNDERSTANDING
p. 316, #10–13

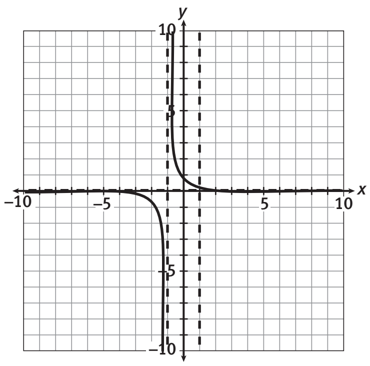
UNIT 5 PRACTICE
p. 327, #45

Answer Graphs for Alternate Examples

Alternate Example 1



Alternate Example 2



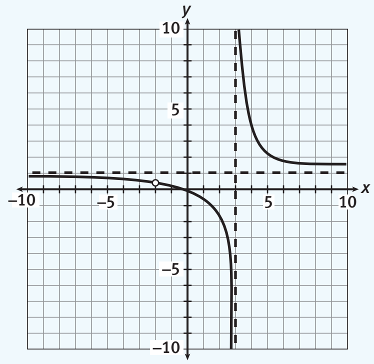
CHECK YOUR UNDERSTANDING

1. $\dfrac{2x+1}{x-2}$, $x \neq -2, 2$
2. $\dfrac{4-x}{x(x+2)}$, $x \neq 0, -4, -2$
3. $\dfrac{x+6}{x-20}$
4. $\dfrac{1}{x-1}$
5. $\dfrac{(x+2)(x-3)}{2(x-2)}$
6. $5(x+2)^2$
7. $(x-3)^2(x+10)$
8. $\dfrac{-x^2-x-2}{(x+3)(x+1)}$
9. $\dfrac{x+2}{(x-4)(x+1)(x-1)}$

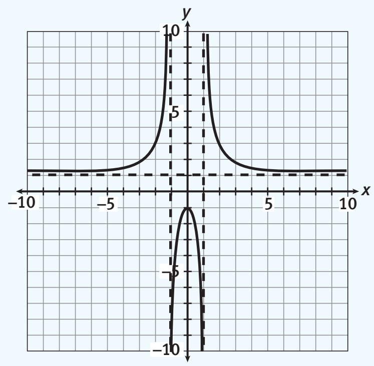
10a. hole: $x = -5$, Vertical Asymptote: $x = 7$, Horizontal Asymptote: $y = 1$

b. hole: $x = -2$, Vertical Asymptote: $x = 2$, Horizontal Asymptote: $y = 0$

11. x-and y-intercept 0, hole: $x = -2$, Vertical Asymptote: $x = 3$, Horizontal Asymptote: $y = 1$



12. y-intercept -1, Vertical Asymptote: $x = -1$, $x = 1$, Horizontal Asymptote: $y = 1$



13. Answers may vary.

ACTIVITY 5.6 continued

Simplifying Rational Expressions
It's All Rational

My Notes

CHECK YOUR UNDERSTANDING

Write your answers on notebook paper or grid paper. Show your work.

Simplify. Identify any restrictions on x.

1. $\dfrac{2x^2+5x+2}{x^2-4}$
2. $\dfrac{16-x^2}{x^3+6x^2+8x}$

Perform the indicated operation. Assume no denominator is equal to zero.

3. $\dfrac{x^2-5x-6}{x^2-12x+36} \cdot \dfrac{x^2-36}{x^2-19x-20}$
4. $\dfrac{2x^2+3x+1}{x^2-1} \cdot \dfrac{2x+1}{4x^2+4x+1}$
5. $\dfrac{3x^2+4x-4}{2x-4} \div \dfrac{9x^2-4}{3x^2-7x-6}$

Find the simplest common denominator.

6. $\dfrac{1}{5x+10}$ and $\dfrac{2}{x^2+4x+4}$
7. $\dfrac{1}{x-3}$, $\dfrac{x}{x^2-6x+9}$, and $\dfrac{2x}{x^2+7x-30}$

Perform the indicated operation. Assume no denominator is equal to zero.

8. $\dfrac{2}{x+3} - \dfrac{x}{x-1}$
9. $\dfrac{2}{x^2-3x-4} - \dfrac{1}{x^2-1}$
10. For each function, identify any vertical asymptotes and horizontal asymptotes.
    a. $f(x) = \dfrac{x^2-25}{x^2-2x-35}$
    b. $f(x) = \dfrac{2x+4}{x^2-4}$

Analyze and graph each rational function.

11. $f(x) = \dfrac{x^2+2x}{x^2-x-6}$
12. $f(x) = \dfrac{x}{x+1} + \dfrac{1}{x-1}$
13. **MATHEMATICAL REFLECTION** What have you learned about simplifying rational expressions and graphing rational functions as a result of this activity?

Rational Equations and Inequalities
A Rational Pastime

ACTIVITY 5.7

SUGGESTED LEARNING STRATEGIES: Marking the Text, Summarize/Paraphrase/Retell, Questioning the Text, Create Representations, Note Taking, Identify a Subtask, Think/Pair/Share

My Notes

Jesse pitches for the baseball team and wants to improve his batting average before the county all-stars are selected. To date, he has 10 hits out of 40 times at the bat.

1. Batting average is the ratio of hits to at-bats. Write a ratio that represents Jesse's current batting average for this season and express the ratio in decimal form.

$\frac{10}{40} = 0.250$

CONNECT TO MEASUREMENT

When a ratio is formed by two quantities with different units, it is also called a *rate*. Batting average is a rate, and even though we call it an average, it does not represent the mean of a set of numbers.

Jesse wants to improve his batting average to at least 0.320. If he gets a hit every time he bats, then his new batting average would be $\frac{10 + x}{40 + x}$ where x is the number of future hits in as many times at-bat.

2. Write an equation to determine how many consecutive hits he needs to bat 0.320.

$\frac{10 + x}{40 + x} = 0.320$

To solve equations like the one you wrote in Item 2, multiply by an expression that eliminates all the denominators.

EXAMPLE 1

Solve $\frac{x^2 - 4}{x + 1} = x + 5$.

Original equation, undefined at $x = -1$ $\qquad \frac{x^2 - 4}{x + 1} = x + 5$

Step 1: *Multiply both sides by $(x + 1)$ to cancel the denominator.* $\qquad (x + 1)\left(\frac{x^2 - 4}{x + 1}\right) = (x + 5)(x + 1)$

Step 2: *Solve for x.*

$$x^2 - 4 = x^2 + 6x + 5$$
$$-4 = 6x + 5$$
$$6x = -9$$
$$x = -1.5$$

Step 3: *Check to see if the original equation is undefined at the solution.*

3. Solve the equation you wrote in Item 2 to find the number of consecutive hits that Jesse needs to increase his batting average.

$x = 4.118$. Jesse would need 5 consecutive hits.

ACTIVITY 5.7 Guided

Rational Equations and Inequalities

Activity Focus

- Solving rational equations and inequalities
- Work problems

Materials

- Graphing calculator

Chunking the Activity

#1–2	#6–8
Example 1	#10–11
#3	#12–13
Example 2–Try These A	Example 3–Try These C
#4–Try These B	

First Paragraph Marking the Text

1 Summarize/Paraphrase/Retell

Second Paragraph Marking the Text This setting of batting average may reveal a possible misconception about the addition of fractions. If Jesse gets 3 hits in his next 4 at-bats, then you would determine his average by $\frac{10 + 3}{40 + 3}$, not by adding the fractions $\frac{10}{40} + \frac{3}{3}$. That's why the expression in this situation is given with "$+ x$" in both the numerator and denominator.

2 Create Representations Check student equations before continuing.

Third Paragraph Summarize/Paraphrase/Retell

EXAMPLE 1 Note Taking

3 Identify a Subtask, Think/Pair/Share, Debriefing

CONNECT TO Measurement: Ratios vs. Rates

There has been much discussion in mathematics education over the difference between ratios and rates. Some definitions and textbooks separate them as distinct terms, while others state that a rate is simply a special type of ratio.

ratio: any comparison of two numbers or measurements
rate: a special ratio in which the two terms have different units

Most mathematicians agree that *ratio* is a more general term that encompasses rates, and is simply a relationship between two quantities. In this activity, *ratio* is used as a generic term, while *rate* can be applied to ratios that involve two different kinds of units.

EXAMPLE 2 Note Taking, This problem is more complicated than Example 1 because it involves subtraction and multiple rational expressions.

TRY THESE A Think/Pair/Share, Group Presentation Have students work in groups and give them the opportunity to check their solutions.

Paragraph Vocabulary Organizer, Interactive Word Wall Extraneous solutions may occur when you multiply or divide both sides of the original equation by the simplest common denominator. The rational equation is transformed into a polynomial equation. Solutions to the resulting polynomial equation are not always solutions to the original rational equation because multiplying by the fraction $\frac{f(x)}{f(x)}$ as you rewrite the equation leads to extraneous solutions when $f(x) = 0$.

4 Identify a Subtask

5 Guess and Check Be sure that students check their solutions in the original equation to find any extraneous solutions.

TRY THESE B Think/Pair/Share, Debriefing

My Notes

SUGGESTED LEARNING STRATEGIES: Note Taking, Think/Pair/Share, Group Presentation, Vocabulary Organizer, Interactive Word Wall, Identify a Subtask, Guess and Check

EXAMPLE 2

Solve $\frac{2}{x} - \frac{1}{x+2} = \frac{3}{x}$.

Original equation, undefined at $x = 0$ and $x = -2$ $\qquad \frac{2}{x} - \frac{1}{x+2} = \frac{3}{x}$

Step 1: *Multiply both sides by $x(x+2)$ to cancel the denominators.* $\qquad x(x+2)\left(\frac{2}{x} - \frac{1}{x+2}\right) = \left(\frac{3}{x}\right)x(x+2)$

Step 2: *Solve for x.*

$$2(x+2) - 1(x) = 3(x+2)$$
$$2x + 4 - x = 3x + 6$$
$$x + 4 = 3x + 6$$
$$-2x = 2$$
$$x = -1$$

Step 3: *Check to see if the original equation is undefined at the solution.*

MATH TIP

When checking your solutions, substitute the solution into the original equation.

TRY THESE A

Solve each equation and check your solution.

a. $\frac{x+4}{x+5} = \frac{3}{5}$

$x = -2.5$

b. $\frac{2x}{x+2} - \frac{x}{x-1} = 1$

$x = \frac{2}{5}$

When solving a rational equation, it is possible to introduce an *extraneous solution*. An **extraneous solution** is not valid in the original equation although it satisfies the polynomial equation that results when you multiply by the simplest common denominator.

4. Solve the equation $\frac{1}{x} - \frac{2x}{x+2} = \frac{x-6}{x(x+2)}$.

$x = 2$

5. Identify any extraneous solutions to the equation in Item 4.

$x = -2$ is the extraneous solution.

TRY THESE B

Solve each equation. Identify any extraneous solutions. Write your answers on notebook paper. Show your work.

a. $\frac{x}{x-1} = \frac{1}{x-1} + \frac{2}{x}$ $\qquad x = 2$; $x = 1$ is an extraneous solution.

b. $\frac{1}{x} - \frac{x-1}{x^2+x} = \frac{x-1}{x+1}$ $\qquad x = 2$; $x = -1$ is an extraneous solution.

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318 SpringBoard® Mathematics with Meaning™ Algebra 2

SUGGESTED LEARNING STRATEGIES: Shared Reading, Questioning the Text, Think/Pair/Share, Simplify the Problem, Create Representations, Graphic Organizer, Identify a Subtask

My Notes

Jesse's coach requires the team to help prepare the baseball diamond at school. Jesse and Cody working together can clean up the infield in 2 h. If Jesse worked alone, it would take him 5 h. To figure out how long it would take Cody to prepare the infield by himself, you must consider the portion of the job that can be completed in 1 h.

6. If Jesse takes 5 hours to complete the job, then what fraction could he complete in 1 hour, assuming he works at an even pace?

$\frac{1}{5}$

7. If it takes Cody t hours to complete the job, then what fraction could he complete in 1 hour assuming he works at an even pace?

$\frac{1}{t}$

8. Write a similar fraction for the amount of work done in 1 hour when both boys work together.

$\frac{1}{2}$

9. Now write an equation using the verbal model below.

Jesse's work in 1 hour	+	Cody's work in 1 hour	=	Together work in 1 hour
$\frac{1}{5}$	+	$\frac{1}{t}$	=	$\frac{1}{2}$

10. Solve the equation you wrote in Item 9 to determine how long it would take Cody to complete the job if he worked alone.

$t = \frac{10}{3}$ or $3\frac{1}{3}$ hours.

11. Garrett has cleaned up the field on his own and it took him 4 hours. How long will it take all three boys working together to prepare the infield for a game?

$\frac{1}{4} + \frac{1}{5} + \frac{1}{\frac{10}{3}} = \frac{1}{t}$

$t = 1\frac{1}{3}$

MINI-LESSON: Alternative Method for Setting Up a Work Problem

This method focuses on the amount of work each person contributes to the completed job. If Jesse does $\frac{1}{5}$ of a job in an hour, then during the 2-hour job, he will do $2\left(\frac{1}{5}\right)$ or $\frac{2}{5}$ of the work. In 2 hours, Cody will do $\frac{2}{t}$ of the job. Together they will complete 1 job.

Jesse's share of the job	+	Cody's share of the job	=	the completed job
$2\left(\frac{1}{5}\right)$	+	$2\left(\frac{1}{t}\right)$	=	1

This equation is equivalent to the one in Item 9.

ACTIVITY 5.7 *Continued*

First Paragraph Shared Reading, Questioning the Text

6-9 These Items introduce a typical work problem. The underlying idea in solving these types of problems is to consider the fraction of a job completed in one hour working alone and working together.

6-8 Think/Pair/Share Students set up fractions for the work done by each student and the combined work done by all.

9 Simplify the Problem, Create Representations, Graphic Organizer This Item uses a verbal model to help students set up their equation. Encourage students to continue to use a verbal model as a guide when solving work problems.

10 Think/Pair/Share This is not a difficult equation to solve. Setting up the equation in Item 9 will be the challenge for most students. Have students present their work on whiteboards. Keep a sample of a work problem posted in the classroom as well.

11 Identify a Subtask, Create Representations, Debriefing This Item extends the work that students did in the previous problems. Because students found Cody's time in Item 10, the variable in this Item will be the total time to finish the job.

Suggested Assignment

CHECK YOUR UNDERSTANDING
p. 322, #1–6

UNIT 5 PRACTICE
p. 327, #46–49

ACTIVITY 5.7 *Continued*

TEACHER TO TEACHER The focus of the second half of this activity is solving rational inequalities. A graphical/numerical approach is introduced that relies on graphing calculator technology.

12 Questioning the Text, Identify a Subtask

13 a Look for a Pattern Make the connection between the graph and the table explicit to students. A more analytical approach is used that relies on analyzing the sign of the rational expression between its zeros and undefined values.

13 b Quickwrite, Debriefing Solutions to these problems are expressed using inequalities. This might be something you need to review with your students. An inequality like $x < 3$ could also be written as $-\infty < x < 3$. Always use a strict inequality when the endpoint is positive or negative infinity.

TECHNOLOGY TIP

- In order to see all of the sign changes, it may be necessary to increment the table by a number smaller than 1.
- Make sure rational expressions are entered into the calculator correctly—the entire denominator and the entire numerator of the rational function must be enclosed in parentheses for the calculator to evaluate the function using the correct order of operations.
- Most graphing calculators will indicate points where a function is undefined by leaving the y-value blank or displaying an error message.
- Try to determine ahead of time a reasonable window that will allow students to see zeros and asymptotes.

SUGGESTED LEARNING STRATEGIES: Questioning the Text, Identify a Subtask, Look for a Pattern, Quickwrite

My Notes

The rational inequality shown below can be solved graphically or numerically.

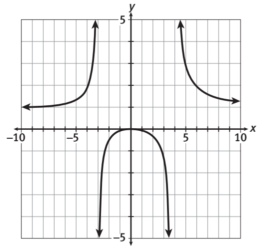
$$\frac{x^2 - 1}{x^2 - x - 12} < 0$$

12. First, factor the left side of the inequality and determine the zeros and the values of x that are not in the domain of the function.

$\frac{(x + 1)(x - 1)}{(x - 4)(x + 3)} < 0$; vertical asymptotes at $x = 4, -3$; zeroes at $x \pm 1$

13. The graph of the left side of the inequality is shown below. The table shows the x- and y-coordinates and the sign of y.

x	y	sign
−5	1.333	+
−4	1.875	+
−3	undefined	
−2	−0.5	−
−1	0	
0	0.083	+
1	0	
2	−0.3	−
3	−1.333	−
4	undefined	
5	3	+



a. Identify the intervals of x where the graph is below the x-axis.

$-3 < x < -1$, $1 < x < 4$

b. Look back to the original inequality. Why would the intervals of x where the graph is below the x-axis be the solutions to the inequality?

The rational expression on the left hand side of the equation is less than zero when the y-values are negative. This happens in quadrants III and IV or when the graph is below the x-axis.

SUGGESTED LEARNING STRATEGIES: Notetaking, Graphic Organizer, Identify a Subtask, Look for a Pattern, Create Representations

My Notes

You can solve rational inequalities without using tables and graphs.

Solving Rational Inequalities

- Write the inequality in factored form.
- Identify the zeros of the numerator and the zeros of the denominator. (Note that the zeros of the denominator are the values where the rational function is not defined.)
- Pick one test value for x that falls between each of the zeros.
- Evaluate the left hand side of the inequality at these values to test the sign of the inequality in each interval and determine the solution.
- State the solution intervals and graph them on the number line.

MATH TIP

To use this method, the inequality must be set up so that one side is a single rational expression and the other side is 0. If it is not, first transform the inequality into that form.

EXAMPLE 3

Solve the inequality $\frac{x^2 - 1}{x^2 - 2x - 8} \le 0$.

Factor: $\frac{(x + 1)(x - 1)}{(x - 4)(x + 2)} \le 0$

Zeros of the numerator at $x = 1$ and -1
Zeros of the denominator (where function is undefined) at $x = -2$ and 4

The zeros, in order from least to greatest are: $-2, -1, 1, 4$
Pick and test one value in each interval: $-3, -1.5, 0, 2$, and 5

For $x = -3$:

$$\frac{(-3 + 1)(-3 - 1)}{(-3 - 4)(-3 + 2)} = \frac{(-2)(-4)}{(-7)(-1)} > 0$$

For $x = -1.5$:

$$\frac{(-1.5 + 1)(-1.5 - 1)}{(-1.5 - 4)(-1.5 + 2)} = \frac{(-0.5)(-2.5)}{(-5.5)(0.5)} < 0$$

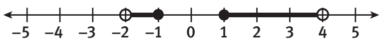
Continue this process and record the results in a table.

interval	$x < -2$	$-2 < x \le -1$	$-1 \le x \le 1$	$1 \le x < 4$	$x > 4$
test value	-3	-0.5	0	2	5
sign	+	−	+	−	+

The solution is the intervals of x where the inequality was less than or equal to 0 (recall the "≤ 0" in the original inequality). So, x-values of the numerator zeros are included in the solution.

Solution intervals: $-2 < x \le -1$ or $1 \le x < 4$

Graph the solution on a number line.



MATH TIP

You do not need to evaluate the inequality completely. Simply plug in the test values and figure out the sign of each factor and then the overall sign of the rational expression.

ACTIVITY 5.7 *Continued*

TEACHER TO TEACHER Solving rational inequalities is a lengthy and time-consuming process when students do not have a graphing calculator. Using a graphic organizer and providing a clear model for students to follow will help them work through these multiple steps.

First Paragraph and Bulleted List Note Taking, Graphic Organizer Remind your students that they do not need to calculate the value of the expression at the test values, just its sign.

Point out the information in the Math Tip about the form of the inequality needed in order to use this method of solution.

EXAMPLE 3 Identify a Subtask, Look for a Pattern, Create Representations, Debriefing The example only shows how to test 2 of the 5 values selected that fall in between the zeros and the values where the function is undefined. You may want to have students practice testing the other values.

Debrief each step by having students compare their answers to the graph of the corresponding rational function on a graphing calculator. They can use that to check their solutions.

TEACHER TO TEACHER There is a reason why the use of test values works to identify the solution sets, after you have displayed all zeros and all values not in the domain of the function. This method of test values works because, by the Intermediate Value Property of Polynomials (deducible from the Fundamental Theorem of Algebra), if a polynomial has values $P(a)$ and $P(b)$ that have different signs, there must be a value c between a and b at which $P(c) = 0$. Consequently, if all the zeros of the numerator polynomial and the denominator polynomial have been located, it follows that the sign cannot change in intervals formed by these values.

ACTIVITY 5.7 *Continued*

TEACHER TO TEACHER While not addressed in a specific example, to solve any rational inequality using the method shown on the previous page, you would first have to manipulate the inequality so it appeared in the form where one side was 0.

Here is an example.

$$\frac{1}{x} - 2 < \frac{1}{x+1}$$

$$\frac{1}{x} - 2 - \frac{1}{x+1} < 0$$

$$\frac{1(x+1) - 2(x)(x+1) - 1(x)}{x(x+1)} < 0$$

$$\frac{-2x^2 - 2x + 1}{x(x+1)} < 0$$

This inequality could now be solved using the methods given in this activity.

TRY THESE C Graphic Organizer, Identify a Subtask, Look for a Pattern, Guess and Check, Create Representations, Group Presentation, Debriefing In Part (b), students must first subtract the two rational expressions on the left side to obtain a single rational expression.

Suggested Assignment

CHECK YOUR UNDERSTANDING
p. 322, #7–9

UNIT 5 PRACTICE
p. 327, #50

CHECK YOUR UNDERSTANDING

1. $x = -5, 2$
2. no solution
3. $x = 4$; $x = 1$ is extraneous.
4. $x = -3$
5. The chemist should add about 28.6 units to the solution.
6. It will take about 1.3 hours.
7. $-3 < x < -1$ or $3 < x < 5$
8. $x \leq -3$ or $-1 < x < 1$
9. Answers will vary.

SUGGESTED LEARNING STRATEGIES: Graphic Organizer, Identify a Subtask, Look for a Pattern, Guess and Check, Create Representations, Group Presentation

My Notes

TRY THESE C

Solve each inequality.

a. $\frac{x^2 - 5x - 6}{x^2 - 4x + 3} \geq 0$

zero at $x = 6$ and -1, undefined at $x = 1$ and 3,
solution: $x \leq -1$ or $1 < x < 3$ or $x \geq 6$

b. $\frac{1}{x} - \frac{2}{x+2} < 0$

zero at $x = 2$, undefined at $x = 0$ and -2,
solution: $-2 < x < 0$ or $x > 2$

CHECK YOUR UNDERSTANDING

Write your answers on notebook paper or grid paper. Show your work.

Solve each equation. Identify any extraneous solutions.

1. $\frac{x+3}{2} = \frac{5}{x}$
2. $\frac{2}{x} + \frac{3}{x} = \frac{5}{x+1}$
3. $\frac{x-3}{x-1} - \frac{2}{x+1} = \frac{x-5}{x^2-1}$
4. $\frac{1}{x-3} = \frac{x}{9-3x}$
5. A chemist has 100 units of a 10% solution and wants to strengthen it to 30%. How much pure chemical should be added to the original solution to achieve the desired concentration?
6. Raj, Ebony, and Jed paint houses during the summer. Raj takes 5 hours to paint a room by himself while it takes Ebony 4 hours and Jed 3 hours. How long will it take them if they work together?

Solve each inequality graphically or numerically.

7. $\frac{x^2 - 9}{x^2 - 4x - 5} < 0$
8. $\frac{1}{x+1} \geq \frac{2}{x-1}$
9. MATHEMATICAL REFLECTION What have you learned about solving rational equations and inequalities as a result of this activity?

Rational Equations and Functions

PLANNING A PROM

Embedded Assessment 2

Use after Activity 5.7.

1. Sketch the graph of the rational function $f(x) = \frac{x-1}{x+1}$. Identify the key features, such as asymptotes and intercepts. Then describe the graph as a transformation of the parent function $f(x) = \frac{1}{x}$.

2. Simplify the rational expression $\frac{x-2}{x+2} - \frac{x+1}{x-1}$. Assume no denominator is zero.

3. Solve the equation $\frac{2}{x} - \frac{2}{x+2} = 8$ in two ways and discuss the advantages and disadvantages of the methods you selected.

4. The prom committee is planning this year's prom. The costs are stated in the table below.

Music	$800
Decorations	$500
Ballroom	$900
Catered Dinner	$25/person

The committee must set aside 10 free prom tickets for a drawing the principal wants to have for students enrolled in Advanced Placement® classes. The ballroom can only hold 300 students, and the prom committee has not decided whether they want to have dinner catered at the prom. Based on past experience, the committee knows students are not willing to pay more than $20 per ticket if food will not be provided and $35 per ticket if food is provided.

Write a proposal for setting the prom ticket price. Be sure to include these items:

- the proposed ticket price
- the number of tickets that must be sold to break even
- the amount of money that will be made
- mathematics to support your reasoning

2 This Item focuses on simplifying a rational expression.

$$\frac{x-2}{x+2} - \frac{x+1}{x-1}$$
$$\frac{(x-2)(x-1) - (x+1)(x+2)}{(x+2)(x-1)}$$
$$\frac{-6x}{(x+2)(x-1)}$$

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Embedded Assessment 2

Activity Focus

- Analyzing and graphing rational functions
- Solving rational equations
- Rational models and applications

Materials

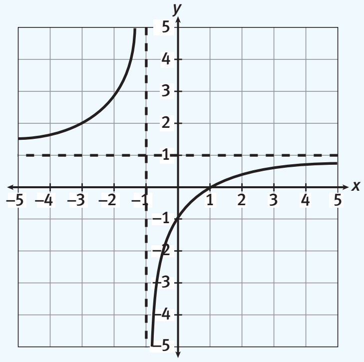
- Graphing calculator

TEACHER TO TEACHER You may wish to assign Item 4 as a take-home quiz to give students the opportunity to plan a thoughtful, well-written response to the Item.

1 This Item asks students to graph a rational function and then describe the graph as a transformation of the parent rational function. Students may do this algebraically using long division or from the graphs they create.

Answer Key

1. x-intercept 1 and y-intercept -1, Vertical Asymptote: $x = -1$, Horizontal Asymptote: $y = 1$. This graph is the parent function stretched vertically by a factor of 2, reflected about the x-axis, and translated 1 unit left and 1 unit up.



TEACHER TO TEACHER You may wish to read through the rubric on the next page with students and discuss the differences in the expectation levels. Make sure students understand the meanings of any terms used.

Unit 5 • Radical and Rational Functions 323

Embedded Assessment 2

3 Students will most likely solve this equation algebraically and graphically using a graphing calculator to find the intersection points.

Solve algebraically.

$$\frac{2}{x} - \frac{2}{x+2} = 8$$

multiply by $x(x + 2)$

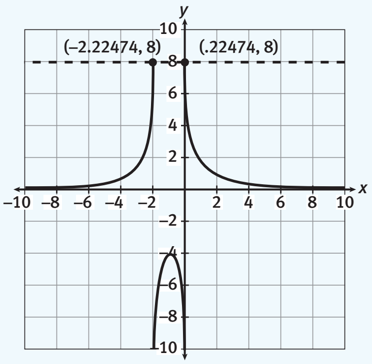
$$2(x + 2) - 2(x) = 8(x)(x + 2)$$

$$4 = 8x^2 + 16x$$

$$8x^2 + 16x - 4 = 0$$

$$x = \frac{-2 \pm \sqrt{6}}{2}$$

Solve graphically using a calculator.



4 Students will need time to read and process this Item and to prepare a response. The Item revisits a context similar to Activity 5.4 Summer Camp, but this time students are planning a prom. This Item requires them to create rational functions from verbal models, solve rational equations/inequalities, graph rational functions, make decisions based on mathematical information and communicate their understanding in written form.

4. Answers may vary but should include the following information. Answers should include multiple representations to justify chosen ticket price.

Embedded Assessment 2

Use after Activity 5.7.

Rational Equations and Functions

PLANNING A PROM

	Exemplary	Proficient	Emerging
Math Knowledge #1, 2, 3, 4	The student: • Identifies four key features of the graph. (1) • Simplifies the rational expression correctly. (2) • Solves the equation correctly in two ways. (3) • Uses the correct mathematics to support his/her reasoning. (4)	The student: • Identifies only three key features of the graph. • Uses the correct method to simplify the expression, but makes a computational error. • Solves the equation correctly in only one way. • The mathematical support contains computational errors.	The student: • Identifies at least one key feature of the graph. • Does not simplify the expression correctly. • Does not solve the equation correctly. • Does not use the correct mathematics to support his/her reasoning.
Problem Solving #4	The student correctly determines the number of tickets that must be sold to break even and the amount of money that will be made for the proposed ticket price. (4)	The student correctly determines the number of tickets that must be sold to break even or the amount of money that will be made for the proposed ticket price.	The student correctly determines neither the number of tickets that must be sold nor the amount of money that will be made.
Representations #1	The student sketches a correct graph. (1)	The student sketches a partially correct graph.	The student sketches an incorrect graph.
Communication #1, 3, 4	The student: • Gives a complete description of the transformation. (1) • Gives a complete discussion of the advantages and disadvantages of the methods selected. (3) • Writes a proposal that includes the four indicated items. (4)	The student: • Gives an incomplete description of the transformation. • Gives an incomplete discussion of the advantages and disadvantages of the methods selected. • Writes a proposal that includes only three indicated items.	The student: • Gives an inaccurate description of the transformation. • Does not discuss the advantages and disadvantages of the methods selected. • Writes a proposal that includes at least one of the indicated items.

Ticket price function without food

$$f(x) = \frac{2200}{x - 10}$$

Ticket price function with food

$$f(x) = \frac{2200 + 25x}{x - 10}$$

The prom committee must sell at least 120 tickets at \$20 to break even if they don't provide food. The prom committee must sell at least 255 tickets at \$35 to break even if they provide food.

If a lower ticket price is selected, then the committee will need to sell more tickets to break even. Evaluate $f(300)$ to determine the minimum per person ticket price. If 300 people attended the prom, then the ticket price without food could be \$7.59 per person and with food it could be \$33.45 per person.

ACTIVITY 5.1

1. What is the inverse of $f(x) = 2x + 7$?

a. $f^{-1}(x) = \frac{x}{2} + 7$ **b.** $f^{-1}(x) = \frac{x-7}{2}$

c. $f^{-1}(x) = 7x + 2$ **d.** $f^{-1}(x) = -2x - 7$

2. The formula $f(x) = \frac{9}{5}x + 32$ converts x degrees Celsius to $f(x)$ degrees Fahrenheit.

a. Find the inverse of $f(x) = \frac{9}{5}x + 32$.

b. Identify 3 ordered pairs (a, b) that satisfy the equation $f(x) = \frac{9}{5}x + 32$. Show that the ordered pairs (b, a) satisfy the equation you found in Part (a).

c. What is the meaning of the ordered pairs (a, b) from part (b) in terms of temperatures? What is the meaning of the ordered pairs (b, a) from part (b) in terms of temperatures?

d. Is there a temperature that has the same numerical value in degrees Fahrenheit and degrees Celsius? Explain your reasoning.

3. What is the inverse of $f(x) = \sqrt{x - 3}$? State the domain and range of the function and its inverse.

4. Which functions are one-to-one?

a. $f(x) = x^2$ **b.** $f(x) = \log_2(x)$

c. $f(x) = x^3 - x$

5. The formula to convert from x in the Kelvin scale to $c(x)$ degrees Celsius is $c(x) = x - 273.15$.

a. Use your knowledge of composition of functions to write a formula to convert Kelvins to degrees Fahrenheit.

b. Use your knowledge of inverse functions to write a formula to convert degrees Fahrenheit to Kelvins.

c. Absolute zero is 0 Kelvin. Convert this temperature to degrees Fahrenheit.

d. Water boils at 212° Fahrenheit. Convert this temperature to Kelvins.

ACTIVITY 5.2

Describe each function as a transformation of $f(x) = \sqrt{x}$. State the domain and range.

6. $f(x) = -4\sqrt{x} + 2$ **7.** $f(x) = -3 + \sqrt{x + 2}$

Graph each function using your knowledge of transformations.

8. $f(x) = \sqrt{x + 4} - 5$ **9.** $f(x) = -3 - 2\sqrt{x - 1}$

Solve for x.

10. $\sqrt{x + 5} = 7$ **11.** $\sqrt{x + 3} = 2x - 9$

12. The radius of a circle is given by the formula $r = \sqrt{\frac{A}{\pi}}$.

a. What is r when $A = 100\pi$?

b. What is r when $A = 100$?

c. What is A when $r = 7$?

ACTIVITY 5.3

Simplify each expression.

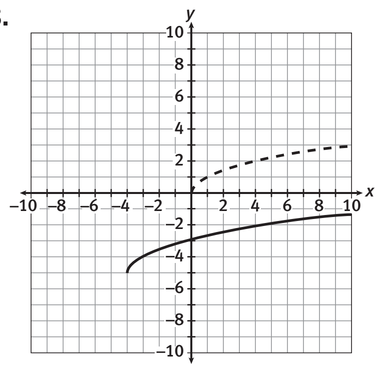
13. $81^{-\frac{1}{4}}$ **14.** $(8x^4)^{\frac{1}{3}}$

15. $6^{\frac{2}{3}} \cdot 36^{\frac{2}{3}}$

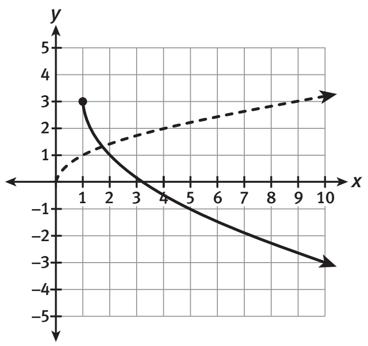
Solve each equation. Assume $x > 0$.

16. $3x^{\frac{1}{4}} = 15$ **17.** $3x^3 + 81 = 0$

8.



9.



UNIT 5 PRACTICE

Activity 5.1

1. B

2a. $f^{-1}(x) = \frac{5}{9}(x - 32)$

b. Answers will vary. One ordered pair solution is shown: (0, 32) is a solution for $f^{-1}(x) = \frac{5}{9}(x - 32)$ because $0 = \frac{5}{9}(32 - 32)$.

c. (0, 32) means 0 degrees Celsius is 32 degrees Fahrenheit. (32, 0) mean 32 degrees Fahrenheit is 0 degrees Celsius. In general, on function f the points are (Celsius, Fahrenheit) and on f^{-1} the points are (Fahrenheit, Celsius).

d. The intersection point of the lines $y = \frac{5}{9}(x - 32)$ and $y = \frac{9}{5}x + 32$ is $(-40, -40)$. Substituting -40 into each temperature conversion formula gives an output of -40.

3. $y = \sqrt{x - 3} \rightarrow x = \sqrt{y - 3}$

f^{-1}: $D = x \geq 0$

$R = y \geq 3$

f: $D = x \geq 3$

$R = y \geq 0$

4. B

5a. $f(x) = \frac{9}{5}(x - 273.15) + 32$ where x is in Kelvin.

b. $k(x) = \frac{5}{9}(x - 32) + 273.15$ where x is degrees Fahrenheit. Function k is the inverse of f.

c. $-459.67°$ F

d. 373.15 K

Activity 5.2

6. Reflection across x-axis, vertical stretch by a factor of 4, vertical translation of 2 up. Domain: $x \geq 0$. Range: $y \leq 2$.

7. Horizontal translation 2 units to the left and vertical translation 3 units up.

Domain: $x \geq -2$. Range $y \geq 3$.

Activity 5.2, 10–12 and Activity 5.3, 13–17: see page 326.

10. $x = 44$ **11.** $x = 6$

12a. $r = 10$ **b.** $r = 5.64$

c. $A = 49\pi$

Activity 5.3

13. $\frac{1}{3}$ **14.** $2x\sqrt[3]{x}$

15. 36 **16.** $x = 625$

17. $x = -3$ **18.** $x - 3$

19. $5\sqrt[4]{125}$

20. $x = 0$ and 1. The solutions are the intersection points of the graphs of $y = x^{\frac{1}{3}}$ and $y = x^{\frac{1}{2}}$.

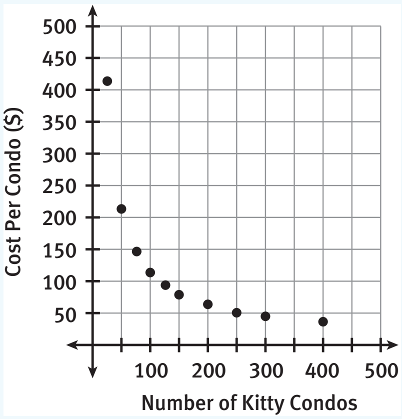
21. $x > 5.196$ or $x < -5.196$

Activity 5.4

22. $f(x) = 10{,}000 + 12x$

23. $c(x) = \dfrac{10{,}000 + 12x}{x}$

24.

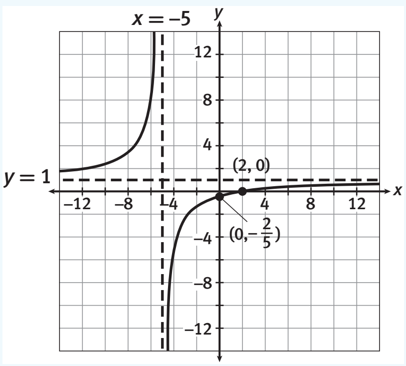


25. \$32

26. Solve the equation $\dfrac{10{,}000 + 12x}{x} = 13$. $x = 10{,}000$

27. C **28.** B **29.** C

30.



UNIT 5 **Practice**

Simplify each expression.

18. $(\sqrt{x} - \sqrt{3})(\sqrt{x} + \sqrt{3})$ **19.** $\dfrac{25}{\sqrt[4]{5}}$

20. Use what you know about the graphs of power functions to solve this equation: $x^{\frac{1}{3}} = x^{\frac{1}{2}}$

21. Solve graphically or analytically: $2x^{\frac{2}{3}} > 6$

ACTIVITY 5.4

KitKat Kondos makes kitty condos. They have \$10,000 in fixed operating costs and each kitty condo costs \$12 to make.

22. Write a function that represents the cost of making x kitty condos.

23. Write a rational function that represents the cost per condo of x kitty condos.

24. Graph the cost per condo function.

25. What is the cost per condo for 500 kitty condos?

26. If the cost per condo was \$13, how many condos did the company make?

27. Which function is an example of a rational function?

a. $f(x) = \frac{x}{3} - 3x^2$ **b.** $f(x) = \frac{x - 3x^2}{2}$

c. $f(x) = \frac{-3x^2}{x - 2}$ **d.** $f(x) = x - 3x^2$

Use $f(x) = \dfrac{x + 2}{2x + 1}$ to answer Items 28–29.

28. What is the vertical asymptote of f?

a. $x = -2$ **b.** $x = -0.5$

c. $x = 0.5$ **d.** $x = 2$

29. What is the horizontal asymptote of f?

a. $y = -2$ **b.** $y = -0.5$

c. $y = 0.5$ **d.** $y = 2$

30. Graph the rational function $f(x) = \frac{x - 2}{x + 5}$.

ACTIVITY 5.5

31. Given the inverse variation $y = \frac{10}{x}$, what is the constant of variation?

a. $k = -10$ **b.** $k = 1$

c. $k = 10$ **d.** $k = 100$

32. If y varies inversely as x, and $y = 8$ when $x = 40$, which equation models this situation?

a. $y = \frac{5}{x}$ **b.** $y = \frac{32}{x}$

c. $y = \frac{48}{x}$ **d.** $y = \frac{320}{x}$

33. Evan's video game scores vary inversely as the time spent playing. If he scores 1000 points after playing for 1 hour, how much will he score after playing 3.5 hours?

34. The number of pages that Emma reads varies directly as the time spent reading. If Emma reads 120 pages in 1.5 hours, how many pages does she read in 45 minutes?

35. Write a function that is $f(x) = \frac{1}{x}$ translated up 2 units and 6 units to the left.

36. Which function is a vertical translation and a vertical stretch of $f(x) = \frac{1}{x}$?

a. $f(x) = \frac{x}{2}$ **b.** $f(x) = \frac{1}{x + 2}$

c. $f(x) = \frac{2}{x} + 1$ **d.** $f(x) = \frac{1}{2} + x$

Activity 5.5

31. C

32. D

33. Evan will score 286 points after 3.5 hours of playing.

34. Emma will read 60 pages in 45 minutes.

35. $f(x) = \dfrac{1}{x + 6} + 2$

36. C

Use $f(x) = \frac{2}{x+1} - 5$ to answer Items 37–38.

37. What is the vertical asymptote of f?

a. $x = -5$ **b.** $x = -1$
c. $x = 1$ **d.** $x = 2$

38. What is vertical stretch of f?

a. 2 **b.** 1
c. -5 **d.** none

39. Graph the rational function $f(x) = \frac{-2}{x+5}$.

ACTIVITY 5.6

40. What are the restrictions on x in the rational expression $\frac{16 - x^2}{4x + 16}$?

a. none **b.** $x \neq 0$
c. $x \neq -4$ **d.** $x \neq \pm 4$

41. Simplify $\frac{16 - x^2}{4x + 16}$. Assume no denominator equals zero.

a. $4 - x$ **b.** $1 - \frac{x}{4}$
c. $\frac{4 - x}{4}$ **d.** $\frac{4 - x}{x + 4}$

42. Divide $\frac{4x + 4}{x^2} \div \frac{x^2 - 1}{x^2 - x}$. Assume no denominator is zero.

a. 4 **b.** $\frac{4}{x}$
c. $4 - x$ **d.** $\frac{4x + 4}{x}$

43. Simplify $\frac{6}{x - 6} + \frac{x}{x + 6}$.

a. -1 **b.** $\frac{x + 6}{x - 6}$
c. $\frac{1}{x - 6}$ **d.** $\frac{x^2 + 36}{(x - 6)(x + 6)}$

44. Simplify $\dfrac{\frac{1}{x-1} - \frac{1}{x}}{\frac{1}{x+1} - \frac{1}{x}}$. Assume no denominator equals 0.

a. 1 **b.** $\frac{1}{x}$
c. $1 - x$ **d.** $\frac{x + 1}{1 - x}$

45. Analyze and graph $f(x) = \frac{x^2 - 36}{x^2 - 5x - 6}$.

ACTIVITY 5.7

46. Solve $\frac{x + 3}{x - 6} = \frac{2}{5}$.

a. $x = -1$ **b.** $x = -9$
c. $x = 9$ **d.** no solution

47. Solve $\frac{2x}{x - 1} + \frac{x - 3}{x - 1} = 2$.

a. $x = 1$ **b.** $x = 0.5$
c. $x = 1, 0.5$ **d.** no solution

48. The resistance, R_P, of a circuit with 2 resistors in parallel is given in Ohms by $\frac{1}{R_P} = \frac{1}{R_1} + \frac{1}{R_2}$. If the resistors come in 5 ohm increments, find a pair of resistors so the circuit has a resistance of 6 ohms.

49. When Joe and Jon work together, they can wire a room in 8 hours. Working alone, Joe needs 12 hours to wire a room. How long would it take Jon working alone to wire a room?

50. Solve each inequality numerically or graphically.

a. $\frac{x + 3}{x^2 - 1} \leq 0$

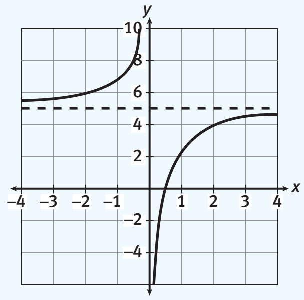
b. $\frac{x^2 - x - 6}{x^2 + 2x - 3} > 0$

UNIT 5 PRACTICE *Continued*

37. B

38. A

39.



Activity 5.6

40. C

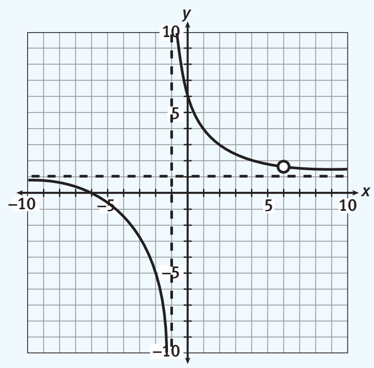
41. C

42. B

43. D

44. D

45. hole: $(6, \frac{12}{7})$,
x-intercept -6,
y-intercept 6,
Vertical Asymptote: $x = -1$,
Horizontal Asymptote: $y = 1$



Activity 5.7

46. b

47. d

48. A 10 ohm and a 15 ohm resistor should be used.

49. It will take Jon 24 hours.

50a. $x \leq -3$ or $-1 < x < 1$

b. $x < -3$ or $-2 < x < 1$ or $x > 3$

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Unit 5 • Radical and Rational Functions 327

Reflection

Student Reflection

Discuss the essential questions with students. Have them share how their understanding of the questions has changed through studying the concepts in the unit.

Review the academic vocabulary. You may want students to revisit the graphic organizers they have completed for academic vocabulary terms and add other notes about their understanding of terms.

Encourage students to evaluate their own learning and to recognize the strategies that work best for them. Help them identify key concepts in the unit and to set goals for addressing their weaknesses and acquiring effective learning strategies.

Teacher Reflection

1. Of the key concepts in the unit, did any present special challenges for students?
2. How will you adjust your future instruction for students/activities?
3. Which strategies were most effective for facilitating student learning?
4. When you teach this unit again, what will you do differently?

UNIT 5

Reflection

An important aspect of growing as a learner is to take the time to reflect on your learning. It is important to think about where you started, what you have accomplished, what helped you learn, and how you will apply your new knowledge in the future. Use notebook paper to record your thinking on the following topics and to identify evidence of your learning.

Essential Questions

1. Review the mathematical concepts and your work in this unit before you write thoughtful responses to the questions below. Support your responses with specific examples from concepts and activities in the unit.
   - Why is it important to consider the domain and range of a function?
   - How are inverse functions useful in everyday life?

Academic Vocabulary

2. Look at the following academic vocabulary words:
   - complex fraction
   - horizontal asymptote
   - inverse variation
   - one-to-one function
   - power function
   - rational exponent
   - rational function
   - vertical asymptote

   Choose three words and explain your understanding of each word and why each is important in your study of math.

Self-Evaluation

3. Look through the activities and Embedded Assessments in this unit. Use a table similar to the one below to list three major concepts in this unit and to rate your understanding of each.

Unit Concepts	Is Your Understanding Strong (S) or Weak (W)?
Concept 1	
Concept 2	
Concept 3	

   a. What will you do to address each weakness?
   b. What strategies or class activities were particularly helpful in learning the concepts you identified as strengths? Give examples to explain.

4. How do the concepts you learned in this unit relate to other math concepts and to the use of mathematics in the real world?

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328 SpringBoard® Mathematics with Meaning™ Algebra 2

Math Standards Review

Unit 5

1. Consider the functions: $c(x) = \frac{1}{\sqrt{x - 4}}$ and $d(x) = 2x$. What is the domain of $c(d(x))$ over the set of real numbers?

A. $\{x: x \in \text{real numbers}\}$

B. $\{x: x > 4\}$

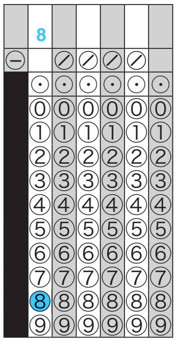
C. $\{x: x > 2\}$

D. $\{x: x > 0\}$

1. Ⓐ Ⓑ Ⓒ Ⓓ

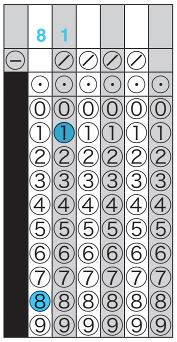
2. What is the solution of $x - 4 = \sqrt{2x}$?

2.



3. What is $(-27)^{\frac{4}{3}}$ in simplified form?

3.



UNIT 5 Math Standards Review

These two pages provide practice with standardized test question formats that are used in many national and state high-stakes tests:

- Multiple choice
- Gridded response
- Constructed response

These items also provide practice with the mathematics content of this unit.

1 Multiple choice
- Composition of functions

2 Gridded response
- Solve equations with radical expressions

3 Gridded response
- Simplify expressions
- Use properties of rational exponents

4 Constructed response
- Graph radical functions

UNIT 5 Math Standards Review

TEACHER TO TEACHER You might read through the extended-response item with students and discuss your expectation levels. Make sure students understand the meanings of any terms used.

Math Standards Review

Unit 5 (continued)

Read
Solve
Explain

4. Given $f(x) = \sqrt{x-1} + 2$ and $g(x) = \sqrt{x}$.

Part A: Describe how to obtain the graph of $f(x)$ from the graph of $g(x)$.

Answer and Explain

a horizontal shift 1 unit to the right and a vertical shift of 2 units up

Part B: State the domain and range of $f(x)$.

Answer and Explain

domain: $x \geq 1$; range: $f(x) \geq 2$

Part C: Graph $f(x)$ and $g(x)$ on the grid provided.