Students continue to expand their repertoire of functions in this unit, while at the same time deepening and extending previously learned concepts such as graphing using transformations, algebraic manipulation of expressions and equations, and inverse relations and functions. The unit opens with an exploration of inverses and composition of functions where students discover that not all functions are invertible. Students then explore radical and rational functions including inverse variation models in a variety of contexts and settings. They gain facility with simplifying expressions and solving equations throughout the unit.

**Academic Vocabulary**

Blackline masters for use in developing students’ vocabulary skills are located at the back of this Teacher’s Edition. Encourage students to explore the meanings of the academic vocabulary words in this unit, using graphic organizers and class discussions to help students understand the key concepts related to the terms. Encourage students to place their vocabulary organizers in their Math notebooks and to revisit these pages to make notes as their understanding of concepts increases.

**Embedded Assessments**

The two Embedded Assessments for this unit follow Activities 5.3 and 5.7.

### Embedded Assessment 1  A Mightier Wind
- Inverse functions
- Composition of functions
- Transformations of \( f(x) = \sqrt{x} \)
- Square root equations
- Rational exponents

### Embedded Assessment 2  Planning a Prom
- Analyzing and graphing rational functions
- Solving rational equations
- Rational models and applications

**AP/College Readiness**

Unit 5 exposes students to power and rational functions in a variety of contextual settings and expands their knowledge about these types of functions, how to manipulate them, graph them, and apply them to physical settings by:
- Giving students a first opportunity to consider functions for which the domain is a subset of the set of real numbers.
- Studying the asymptotic behavior of rational functions and laying a foundation for understanding limits.
- Providing practice simplifying complicated expressions and solving a variety of equations students are likely to encounter in AP courses.
- Using technology as a tool to explore concepts and relationships and well as solve problems.
- Exploring functions numerically, graphically, verbally, and algebraically.
- Creating and using mathematical models based on written descriptions and collected data.
Suggested Pacing
The following table provides suggestions for pacing either a 45-minute period or a block schedule class of 90 minutes. Space is left for you to write your own pacing guidelines based on your experiences in using the materials.

<table>
<thead>
<tr>
<th>Activity</th>
<th>45-Minute Period</th>
<th>90-Minute Period</th>
<th>Comments on Pacing</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unit Overview</td>
<td>1/2</td>
<td>1/4</td>
<td></td>
</tr>
<tr>
<td>Activity 5.1</td>
<td>3</td>
<td>1 1/2</td>
<td></td>
</tr>
<tr>
<td>Activity 5.2</td>
<td>3</td>
<td>1 1/2</td>
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<tr>
<td>Activity 5.3</td>
<td>4</td>
<td>2</td>
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<tr>
<td>Embedded Assessment 1</td>
<td>1</td>
<td>1/2</td>
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</tr>
<tr>
<td>Activity 5.4</td>
<td>4</td>
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<td>Activity 5.5</td>
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</tr>
<tr>
<td>Activity 5.6</td>
<td>4</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>Activity 5.7</td>
<td>3</td>
<td>1 1/2</td>
<td></td>
</tr>
<tr>
<td>Embedded Assessment 2</td>
<td>1</td>
<td>1/2</td>
<td></td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>26 1/2</strong></td>
<td><strong>13 1/4</strong></td>
<td></td>
</tr>
</tbody>
</table>

Unit Practice
Practice Problems for each activity in the unit appear at the end of the unit.

Math Standards Review
To help accustom students to the formats and types of questions they may encounter on high stakes tests, additional problems are provided at the end of the unit. These problems are constructed for multiple choice, short response, extended response, and gridded responses.
Unit Overview
In this unit, you will extend your study of functions to radical, rational, and inverse functions and the composition of functions. You will solve rational equations and inequalities as well as equations with rational exponents.

Unit 5 Academic Vocabulary
Add these words and others that you encounter in this unit to your vocabulary notebook.
- complex fraction
- horizontal asymptote
- inverse variation
- one-to-one function
- power function
- rational exponent
- rational function
- vertical asymptote

Essential Questions
Why is it important to consider the domain and range of a function?
How are inverse functions useful in everyday life?

Embedded Assessments
This unit has two embedded assessments, following Activities 5.3 and 5.7. These assessments will allow you to demonstrate your understanding of inverse functions, the composition of functions, and solving and graphing radical and rational equations.

Embedded Assessment 1
Square Root Expressions, Equations and Functions p. 291

Embedded Assessment 2
Rational Equations and Functions p. 323

Materials
- Graphing calculator

Academic Vocabulary
As students develop fluency with new terms, encourage them to use precise mathematical language in discussions and writing. Continue to monitor their vocabulary logs for completeness.

Embedded Assessments
There are two embedded assessments in this unit with evaluation rubrics. You may want to review skills needed for the assessment with students prior to the beginning of their work.
Write your answers on notebook paper or grid paper. Show your work.

1. What values are not possible for the variable \( x \) in each expression below? Explain your reasoning.
   a. \( \frac{2}{x} \)
   b. \( \frac{2}{x - 1} \)

2. Perform the indicated operation.
   a. \( \frac{2x}{5} - \frac{3x}{10} \)
   b. \( \frac{2x + 1}{x + 3} + \frac{4x - 3}{x + 3} \)
   c. \( \frac{2}{x} + \frac{5}{x + 1} \)

3. Simplify each monomial.
   a. \( (2x^2y^3)(3xy^5) \)
   b. \( (4ab^3)^2 \)

Factor each expression in Items 4–5.

4. \( 81x^2 - 25 \)
5. \( 2x^3 - 5x - 3 \)
6. Simplify \( \sqrt{128x^2} \).
7. Find the composition \( f(g(x)) \) if \( f(x) = 5x - 4 \) and \( g(x) = 2x \).
8. Which of the following is the inverse of \( h(x) = 3x - 7 \)?
   a. \( 7 - 3x \)
   b. \( 3x + 7 \)
   c. \( \frac{x + 7}{3} \)
   d. \( \frac{1}{3x - 7} \)
Composition of Functions
Code Breakers

The use of cryptography goes back to ancient times. In ancient Greece, Spartan generals exchanged messages by wrapping them around a rod called a scytale and writing a message on the adjoining edges. The Roman general and statesman Julius Caesar used a transposition cipher that translated letters three places forward in the alphabet. For example, the word CAT was encoded as FDW.

In modern times, cryptography was used to secure electronic communications. Soon after Samuel F.B. Morse invented the telegraph in 1844, its users began to encode the messages with a secret code, so that only the intended recipient could decode them. During World War II, British and Polish cryptanalysts used computers to break the German Enigma code so that secret messages could be deciphered.

Many young children practice a form of cryptography when writing notes in secret codes. The message below is written in a secret code.

1. Try to decipher the seven-letter word coded above. The word is INVERSE.

2. What do you need to decipher the seven-letter word? A key or cipher is needed to decode the message.

CONNECT TO SCIENCE
Cryptography is the science of code-making (encoding) and cryptanalysis is the science of code-breaking (decoding).

CONNECT TO COMPUTING
Modern computers have completely changed the science of cryptography. Before computers, cryptography was limited to two basic types: transposition, rearranging the letters in a message, and substitution, replacing one letter with another. The most sophisticated pre-computer codes used five or six operations. Computers can now use thousands of complex algebraic operations to encrypt messages.

ACTIVITY 5.1 Investigative

Composition of Functions

Activity Focus
• Inverse functions and relations
• Composition of functions
• One-to-one functions
• Restricted domain and range

Materials
• Graphing calculator

Chunking the Activity
#1–2 #12 #18–20
#3–4 #13 #21
#5–6 #14–16 #22–23
#7–11 #17 #24–26

TEACHER TO TEACHER
You can add visual interest by presenting pictures of a scytale, Julius Caesar, or the Braille alphabet as students read the introduction.

First Four Paragraphs
In the context of cryptography, students investigate inverse relations and functions, and one-to-one functions numerically, algebraically, and graphically. Linear functions are reviewed and then the properties are extended to all functions.

1. Look for a Pattern The cipher used to encode the message, called the Pigpen Cipher, is shown at the bottom of the next page. Allow students time to try to decipher the message before giving them the code.

2. Quickwrite Based on the introduction, students should ask for a key or a code relating the letters and symbols.

CONNECT TO Cryptography
Student projects (such as creating a scytale) or student reports could be developed from the introduction. The first documented examples of cryptography date back to 1900 BCE. Julius Caesar, Sir Francis Bacon, and Thomas Jefferson are well-known users of cryptography. A form of Julius Caesar’s code is used in Item 4. The Braille alphabet and the use of quilts as a means of protecting the Underground Railroad during the Civil War could also be investigated in student reports. Students may be familiar with best-selling novels and popular movies that use cryptography as a central theme. Sarah Flannery’s book In Code: A Mathematical Adventure might also be of interest to students.
### Activity 5.1 Continued

3. Look for a Pattern, Quickwrite
   Given a little time, most students will decipher MATH.

4. Look for a Pattern, Create Representations, Debriefing
   Given that the first two words were academic words, some students are likely to discover the shift of 2 letters and decipher SCHOOL.

**Paragraphs and Tables**

*Marking the Text, Summarize/Paraphrase/Retell*

Students should have the opportunity to decipher the messages without prompting. Some students will be able to decipher each message. They will find it much more rewarding if they can discover the code on their own. To make sure that students don’t look ahead, you may want to distribute a copy of only Items 1–4. Do not spend too much time on these early Items. They are designed only to present a setting in which to study the mathematics.

The code $A = 1, B = 2$, and so on is used throughout the rest of this activity to convert letters to numbers.

**Letter-to-Number Codes**

<table>
<thead>
<tr>
<th>Letter</th>
<th>Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1</td>
</tr>
<tr>
<td>B</td>
<td>2</td>
</tr>
<tr>
<td>C</td>
<td>3</td>
</tr>
<tr>
<td>D</td>
<td>4</td>
</tr>
<tr>
<td>E</td>
<td>5</td>
</tr>
<tr>
<td>F</td>
<td>6</td>
</tr>
<tr>
<td>G</td>
<td>7</td>
</tr>
<tr>
<td>H</td>
<td>8</td>
</tr>
<tr>
<td>I</td>
<td>9</td>
</tr>
<tr>
<td>J</td>
<td>10</td>
</tr>
<tr>
<td>K</td>
<td>11</td>
</tr>
<tr>
<td>L</td>
<td>12</td>
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<tr>
<td>M</td>
<td>13</td>
</tr>
<tr>
<td>N</td>
<td>14</td>
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<td>O</td>
<td>15</td>
</tr>
<tr>
<td>P</td>
<td>16</td>
</tr>
<tr>
<td>Q</td>
<td>17</td>
</tr>
<tr>
<td>R</td>
<td>18</td>
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<tr>
<td>S</td>
<td>19</td>
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<tr>
<td>T</td>
<td>20</td>
</tr>
<tr>
<td>U</td>
<td>21</td>
</tr>
<tr>
<td>V</td>
<td>22</td>
</tr>
<tr>
<td>W</td>
<td>23</td>
</tr>
<tr>
<td>X</td>
<td>24</td>
</tr>
<tr>
<td>Y</td>
<td>25</td>
</tr>
<tr>
<td>Z</td>
<td>26</td>
</tr>
</tbody>
</table>

### Composition of Functions

**Activity** 5.1

**My Notes**

3. The following message uses a numerical code. Can you decode the four-letter word? Explain how you know.

   13  1  20  8

   The word is MATH. It uses the letter-to-number code.

4. What is this six-letter word?

   21  5  10  17  17  14

   The word is SCHOOL.

In Item 3, a single function was used to encode a word. The function assigned each letter to the number representing its position in the alphabet.

In Item 4, two functions were used to encode a word. The first function assigned each letter to the number representing its position in the alphabet $x$ and then the function $f(x) = x + 2$ was used to encode the message further as shown in the table.

<table>
<thead>
<tr>
<th>LETTER</th>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>S</td>
<td>19</td>
<td>21</td>
</tr>
<tr>
<td>C</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>H</td>
<td>8</td>
<td>10</td>
</tr>
<tr>
<td>O</td>
<td>15</td>
<td>17</td>
</tr>
<tr>
<td>L</td>
<td>12</td>
<td>14</td>
</tr>
</tbody>
</table>

### Connect to the Pigpen Cipher

This code may be unfamiliar to most students. The cipher shown below illustrates the possibility of encoding a message using symbols rather than numbers or other letters.

```
A B C
D E F
G H I
J K L
M N O
P Q R
S T U
X Y Z
```

Using this code, the letter $J$ would be
SUGGESTED LEARNING STRATEGIES: Create Representations, Look for a Pattern, Work Backward

5. Write a function $g$ that could decode the message in Item 4 and use it to complete the table below.

$$g(x) = x - 2$$

<table>
<thead>
<tr>
<th>$x$</th>
<th>$g(x)$</th>
<th>LETTER</th>
</tr>
</thead>
<tbody>
<tr>
<td>21</td>
<td>19</td>
<td>S</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>C</td>
</tr>
<tr>
<td>10</td>
<td>8</td>
<td>H</td>
</tr>
<tr>
<td>17</td>
<td>15</td>
<td>O</td>
</tr>
<tr>
<td>14</td>
<td>12</td>
<td>L</td>
</tr>
</tbody>
</table>

6. Try to decipher the more difficult message below. First, each letter in the message was assigned a number based on its position in the alphabet, and then another function encoded the message further.

20 $-1$ 50 8 11 50

The word is HARDER.

7. The encoding function for Item 6 is $f(x) = 3x - 4$.

Write a decoding function $g$ and complete the table below.

$$g(x) = \frac{x + 4}{3}$$

<table>
<thead>
<tr>
<th>$x$</th>
<th>$g(x)$</th>
<th>LETTER</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>8</td>
<td>H</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
<td>A</td>
</tr>
<tr>
<td>50</td>
<td>18</td>
<td>R</td>
</tr>
<tr>
<td>8</td>
<td>4</td>
<td>D</td>
</tr>
<tr>
<td>11</td>
<td>5</td>
<td>E</td>
</tr>
<tr>
<td>50</td>
<td>18</td>
<td>R</td>
</tr>
</tbody>
</table>
Paragraph Interactive Word Wall, Vocabulary Organizer

9 Create Representations, Simplify the Problem It is not enough to show that the definition of inverse function holds true for a single value of x (that is, to show that f(g(3)) = 3 and g(f(3)) = 3). It is also not sufficient to show that only one of the two parts of the definition is true for all x. Try a numerical approach for students who are having difficulty. For example,

\[
\begin{align*}
1 & \rightarrow f \rightarrow 1 & g \rightarrow 1 \\
1 & \rightarrow g \rightarrow \frac{5}{3} & f \rightarrow 1
\end{align*}
\]

10-11 Think/Pair/Share, Debriefing These Items emphasize the use of inverse notation.

Suggested Assignment
CHECK YOUR UNDERSTANDING p. 276, #1a
UNIT 5 PRACTICE p. 325, #1–2

ACTIVITY 5.1 Continued
Composition of Functions
Code Breakers

SUGGESTED LEARNING STRATEGIES: Guess and Check, Interactive Word Wall, Vocabulary Organizer, Create Representations, Simplify the Problem, Think/Pair/Share

8. Verify that your function works by encoding the letter C = 3 with f and then decoding it by using g.
   \[ f(3) = 5 \]
   \[ g(5) = 3 \]
   Recall that functions f and g are called inverse functions if and only if
   \[ f(g(x)) = x \text{ for all } x \text{ in the domain of } g \text{ and } g(f(x)) = x \text{ for all } x \text{ in the domain of } f. \]

9. Use the definition of inverse functions to show that the encoding function \( f(x) = 3x - 4 \) and its decoding function \( g \) are inverses.
   If the functions are inverses then their composition will equal \( x \).
   \[ f(g(x)) = 3 \left(\frac{x + 4}{3}\right) - 4 = x \]
   \[ g(f(x)) = \frac{3x - 4 + 4}{3} = x \]

10. What is \( f^{-1} \) for the function \( f(x) = x + 2? \)
   \[ f^{-1}(x) = x - 2 \]

11. What is \( f^{-1} \) for the function \( f(x) = 3x - 4? \)
   \[ f^{-1}(x) = \frac{x + 4}{3} \]

CONNECT TO AP

One way to think of functions and their inverses is to consider that a function and its inverse “undo” each other. This leads students to the idea that the composition of a function and its inverse leads to the identity function. For example, \( f(x) = 2x - 3 \) multiplies x by 2 and subtracts 3. The inverse function \( g(x) \) would add 3 to \( x \) and divide the result by 2 (i.e., \( g(x) = \frac{x + 3}{2} \)). This leads to \( f(g(x)) = g(f(x)) = x \).

Similarly, inverse relationships will be considered when studying exponential and logarithmic functions and trigonometric functions. In calculus, students will discover a similar relationship between a particular function’s derivative and a related anti-derivative.
So far, the functions in this activity have been linear functions. Other types of functions also have inverses.

12. The graph of \( f(x) = \sqrt{x} \) is shown.

a. List four points on the graph of \( f \) and four points on its inverse.
   - Points on \( f : (0, 0), (1, 1), (4, 2), (9, 3) \), and points on the inverse \((0, 0), (1, 1), (2, 9), (3, 4)\).

b. Use the points from Part (a) to graph the inverse of \( f \).

math tip
Recall that if \((a, b)\) was a point on the graph of a function then \((b, a)\) must be a point on the graph of the inverse of the function.

UNIT 5: RADICAL AND RATIONAL FUNCTIONS

CONNECT TO Inverses
You may want to review the definition of and properties of inverse functions that students learned in Activity 1.5. If needed, you can also review the process to find the inverse of a relation algebraically.

\[
f(x) = x + 2
\]
\[
y = x + 2 \quad \text{Let } y \text{ represent } f(x).
\]
\[
x = y + 2 \quad \text{Interchange } x \text{ and } y \text{ to form the inverse relationship.}
\]
\[
y = x - 2 \quad \text{Solve for } y \text{ to find the inverse. (Assume the inverse is a function.)}
\]
\[
f^{-1}(x) = x - 2 \quad \text{This is the inverse function.}
\]

TECHNOLOGY Tip
Students may graph functions with a restricted domain on their calculator. Using Boolean algebra, students can enter \( y = \frac{x}{x - 2} \) for \( x \neq 2 \).
ACTIVITY 5.1 Continued

First Paragraph  Interactive Word Wall, Vocabulary Organizer

12 a Create Representations
Students should find a few points on g and then use those to graph the inverse.

12 b Quickwrite The inverse of g is not a function. Acceptable student answers may include the following:
- A reference to the vertical line test
- For any x > 4, there are two corresponding y-values
- An example, such as: if x = 5, then y = 1 or y = -1

12 c Look for a Pattern At this point, it is expected that students will understand that they do not need to see the graph of the inverse in order to determine whether it is a function. Students should have discovered a horizontal line test. If every horizontal line intersects the graph of the function in at most one point, then the function is one-to-one (and its inverse will be a function). These definitions are presented on the next page of the student edition.

12 d Work Backward, Debriefing x = y^2 + 4 is the inverse relation of g. This statement can also be written as y = ±√x - 4.

13 Use the quadratic function g graphed below.

a. Graph the inverse of g.

b. Is the inverse of g a function? Explain your reasoning.
   Explanations will vary. Sample response: The inverse of g is not a function because it has 2 y-values for at least one x-value.

c. What characteristic of the graph of a function can you use to determine whether its inverse relation is a function?
   Answers will vary. Sample response: If a function has 2 x-values for at least one y-value then its inverse relation will not be a function.

d. The quadratic function shown in the graph is g(x) = x^2 + 4. Find an equation for the inverse relation of this function.
   The equation is x = y^2 + 4, or y^2 = x - 4; y = ±√x - 4 for x ≥ 4.

MATH TERMS
A relation is a set of ordered pairs that may or may not be defined by a rule. Not all relations are functions, but all functions are relations.
A function is defined as **one-to-one** if, for each number in the range of the function, there is exactly one corresponding number in the domain of the function.

14. Is $g$ from Item 13 a one-to-one function? Explain. According to the definition, the function $g$ is not one-to-one because each $y$ (except $y = 4$) has 2 different $x$-values associated with it.

15. What do you know about a function whose inverse relation is a function? That function will have to be one-to-one.

16. Determine whether each type of function will always have an inverse that is a function. Explain your reasoning.
   a. Linear function
      Linear functions ($m \neq 0$) are one-to-one, so they will always have an inverse that is a function.
   b. Quadratic function
      Quadratic functions are not one-to-one, so they will never have an inverse that is a function.

17. Investigate the cubic function $h(x) = x^3 - 6x^2 + 8x + 5$.
   a. Use your calculator to graph $h$ in the viewing window $[-10, 16]$ by $[-10, 16]$ and sketch the results in the My Notes section.
   b. Explain whether or not $h$ is a one-to-one function.
      The function is not one-to-one. For example, there are 3 points that contain $y = 6$.
ACTIVITY 5.1 Continued

Look for a Pattern, Debriefing
Students may quickly answer “Yes” to Part (c) without investigating additional cubic functions. Part (d) may help these students to catch their error.

Suggested Assignment
CHECK YOUR UNDERSTANDING p. 276, #1b, 2–3
UNIT 5 PRACTICE p. 325, #3–4

On this day, students return to the context of cryptography. Students should be familiar with the mechanics of working with composite functions—if not, review content from Activity 1.4. The context will help students improve their intuitive understanding of composite functions. They will also encode messages with functions that are not one-to-one to observe the difficulties of decoding these messages. Students should enjoy the challenges of this section, but it can be omitted if necessary due to time constraints.

Create Representations

Create Representations

Create Representations, Quickwrite Instead of first encoding the numbers and then encoding them a second time, composition allows the two processes to be done in a single step. Students will realize how quickly the composite function $h$ does the work of the other two functions by using it to encode the original message.

17. (continued)
   c–d. Look for a Pattern, Debriefing
   Students may quickly answer “Yes” to Part (c) without investigating additional cubic functions. Part (d) may help these students to catch their error.

SUGGESTED LEARNING STRATEGIES: Look for a Pattern, Create Representations, Quickwrite

17. (continued)
   c. Is your answer in Part (b) true for all cubic functions? Explain.
   No. $y = x^3$ is a one-to-one cubic function.
   
   d. Does a cubic function always have an inverse that is a function? Explain your reasoning.
   No. Some cubic functions are one-to-one and some are not.

Three students are encoding and decoding messages that begin with the numerical code used in Item 3. Suppose that the message HELLO is to be passed from one student to a second student and then on to a third student. Use this information for Items 18–24.

18. The first student translates HELLO to numbers and then encodes it with the function $f(x) = 2x^3 - 12$. What encoded message will the second student receive?

<table>
<thead>
<tr>
<th>H</th>
<th>E</th>
<th>L</th>
<th>L</th>
<th>O</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>5</td>
<td>12</td>
<td>12</td>
<td>15</td>
</tr>
<tr>
<td>-4</td>
<td>2</td>
<td>-12</td>
<td>-12</td>
<td>-18</td>
</tr>
</tbody>
</table>

19. After receiving the encoded message, the second student encodes it again, using the function $g(x) = -x + 9$. What encoded message will the third student receive?

| -4 | 2 | -12| -12| -18|
| 13 | 7 | 21| 21| 27|

20. Let $h$ be the composite function $h(x) = g(f(x))$.

   a. Write a rule for $h$.
   
   $h(x) = g(f(x)) = g(-2x^3 + 12) = -(-2x + 12) + 9$  
   $\quad \quad \quad \quad \quad = 2x - 3$

   b. Explain how $h$ relates to the encoding of the message HELLO.
   Answers will vary. Sample response: It provides a quicker method for encoding the two-step message. Instead of substituting a value into $f$ and then taking that result and substituting it into $g$, you only need to substitute and evaluate once when encoding the message with $h$. 
21. The third student receives another message shown below. What does this message say?

\[
\begin{array}{cccccccc}
27 & 25 & 7 & 37 & 27 & 25 & 7 \\
15 & 14 & 6 & 20 & 15 & 15 & 14 & 6 \\
\end{array}
\]

The message says ONE TO ONE. \( h^{-1}(x) = \frac{x + 3}{2} \)

22. Suppose that the message above had been encoded using the composite function \( k(x) = f(g(x)) \). How would the message have been encoded?

Answers will vary. Sample response: The numbers were substituted into \( g \) and the output was then substituted into \( f \).

\[ K(x) = -2(-x + 9) + 12 = 2x - 6; \]

code: 24 22 4 34 24 22 4

23. Can the rule found in Item 20(a) also be used for \( k \)? Explain.

Answer will vary. Sample response: No. Composition of functions is not commutative. That is in general, \( g(f(x)) \neq f(g(x)) \).

24. Suppose that the third student is given \( f \) and \( g \), and the encoded message. What additional information would help this student to decode the message more efficiently? Explain.

Answers will vary. Sample response: They would need to know the order in which the two functions were composed.

25. Another message is encoded twice. First, the message is encoded with the function \( f(x) = 2x - 29 \), and then it is encoded with the function \( g(x) = x^2 \).

a. Encode the word GRAPH using the composite function.

GRAPH is encoded as \( f(g(x)) = (2x - 29)^2 \) and is shown below.

\[
\begin{array}{cccc}
g & r & a & p \\
225 & 49 & 725 & 9 \\
\end{array}
\]

b. Decode the following message.

\[
\begin{array}{cccccccc}
1 & 121 & 1 & 361 & 121 & 1 & 361 \\
\end{array}
\]

The inverse of the composite function is \( y = \frac{29 \pm \sqrt{x}}{2} \) and is shown below.

\[
\begin{array}{cccccccc}
1 & 1 & 121 & 1 & 361 & 121 & 1 & 361 \\
14 or 15 & 9 or 20 & 14 or 15 & 9 or 24 & 14 or 15 & 9 or 24 & 14 or 15 & 5 or 24 \\
N or O & N or O & N or O & E or X & N or O & N or O & E or X & \\
\end{array}
\]

ACTIVITY 5.1 Continued

21. Work Backward, Create Representations Some students may decode the message as a two-step process. They should be encouraged to use the inverse of \( h \) so that the message can be decoded in a single step.

22. Quickwrite The second student passed the message to the first student, who then passed the message on to the third student.

22. Quickwrite Students should realize that \( k \) and \( h \) are not the same functions, and that, in general, the operation of composition of functions is not commutative.

24. Summarize/Paraphrase/Retell, Quickwrite The message could be decoded more efficiently if the third student knew which function was first used to encode the message. Without this information, a student would have to find the inverses of both \( f \circ g \) and \( g \circ f \). After decoding the message both ways, the student would have to choose the message that makes more sense.

25. Work Backward, Look for a Pattern, Debriefing Because the composite encoding function was not one-to-one, each decoding input has two possible outputs. The message that was actually encoded was NOT ONE TO ONE.
Think/Pair/Share, Quickwrite, Debriefing  Because \( h \) is a function, it can be used to encode messages, as in Item 25. For every letter in a given message, there will be exactly one number in the coded message that is associated with each letter. Since \( h \) is not one-to-one, its inverse is not a function. Therefore, the inverse cannot be used to decode messages because some of the numbers in a given encoded message will have two letters associated with the numbers.

Suggested Assignment
CHECK YOUR UNDERSTANDING
p. 276, #4–5
UNIT 5 PRACTICE
p. 325, #5

CHECK YOUR UNDERSTANDING

1a. \( f^{-1}(x) = \sqrt{x + 6} \), Domain and range of \( f \) and \( f^{-1} \) are both all real numbers.

b. \( f^{-1}(x) = \left( \frac{x}{2} \right)^2 + 5 \), for \( x \geq 0 \).
   Domain of \( f \): \( x \geq 5 \).
   Range of \( f \): \( y \geq 0 \).
   Domain of \( f^{-1} \): \( x \geq 0 \).
   Range of \( f^{-1} \): \( y \geq 5 \)

2. The function \( f(x) = 10^x \) has an inverse function because it is a one-to-one function. The function \( f(x) = x^2 - 10 \) has no inverse function because it is not a one-to-one function. The table does not represent a function that has an inverse because there are two \( x \)-values (−3 and 0) for a single \( y \)-value (−1).

3a. \( g(f(x)) = (2x - 3)^2 - 2(2x - 3) - 8 = 4x^2 - 16x + 7 \)

b. The inverse is not a function because quadratic functions are not one-to-one.

4a. Sometimes, when \( m \neq 0 \).

b. Never, because quadratic functions are not one-to-one.

c. Sometimes, when the cubic function is one-to-one.

5. Answers may vary. Sample answer: The graph of a quadratic function shows that there are two domain values for a single range value, so the function is not one-to-one.
**Suppose the hull speed in knots $H$ of a sailboat is given by the function $H(x) = 1.34\sqrt{x}$, where $x$ is the length of the boat in feet at the waterline.**

1. The hull speed function is a transformation of the parent square root function $f(x) = \sqrt{x}$.
   - Graph $H$ and $f$ on the same axes. How do these graphs compare to each other?
     
     $H$ is a vertical stretch of $f$. Both graphs contain the point $(0, 0)$.

   ![Graph of $H$ and $f$](image)

   - What are the domain and the range of $f$?
     The domain of $f$ is $x \geq 0$. The range of $f$ is $y \geq 0$.

   - What are the domain and range of $H$?
     The domain and range of $H$ are the same as $f$.

2. Explain how you could use transformations of the graph of $f(x) = \sqrt{x}$ to graph $g(x) = 2\sqrt{x}$.
   Multiply the $y$-coordinate of the points on the graph of $f(x) = \sqrt{x}$ by 2.

   This will vertically stretch the parent graph by a factor of 2.

**Math Tip**

To graph the parent square root function, use key points with $x$-values that are perfect squares, such as 0, 1, 4, and 9.

**CONNECT TO SAILING**

The speed of a boat is measured in knots (nautical miles per hour). The distance it travels in water is measured in nautical miles. A nautical mile is equal to 1.15 statute miles.

**Activity Focus**
- Transformations of $f(x) = \sqrt{x}$
- Square root equations
- Extraneous solutions

**Materials**
- Graphing or scientific calculator

**Chunking the Activity**

1. **Activating Prior Knowledge, Create Representations, Quickwrite** Emphasize plotting key points for the parent square root function, such as $(0, 0)$, $(1, 1)$, $(4, 2)$, and $(9, 3)$ when sketching graphs.

   Students could use a calculator to make their graphs if needed.

   Transformations should already be familiar to students, although this will be the first parent function whose domain is not all real numbers. Make sure to reinforce proper vocabulary when students write about how the two graphs compare.

2. **Quickwrite, Debriefing** The vertical stretch factor of 2 implies that each $y$-value of the parent function will be multiplied by 2 to produce the $y$-values for $g$. At this point, students should be proficient at articulating the effect of a vertical stretch.

**CONNECT TO Navigation**

Historically, a nautical mile is the length of an arc equal to 1 minute ($\frac{1}{60}$ of a degree) along a meridian of Earth. The standard used today is 1 nautical mile (NM) = 1852 m (approximately 1.15 miles). The term knot (1 knot = 1 nautical mile per hour) derives from the ‘log line’ used to measure the speed of a ship before electronic devices were available. A log (wedge-shaped piece of wood) tied to a length of knotted rope was thrown overboard. Knots were tied at even spaces along the line at 47 ft 3 in. intervals. Sailors counted the number of knots that passed over the edge in a 28-second interval.
The definitions for the transformations on this page should be familiar to students and are merely provided for reference. You should not need to re-teach these terms or their effect on graphs of functions. This activity merely extends the use of transformation graphing to a different type of parent function.

3-5 Give students time to work these items in their groups.

3 Quickwrite

4 Create Representations
Check to see that student graphs are accurate by using the key points. Students should not be making a table of values to graph g. Instead they should simply plot the key points on the graph of f translated 3 units to the right and then sketch in the curve.

5 Think/Pair/Share, Debriefing

5 Think/Pair/Share, Look for a Pattern, Quickwrite Check to see that students are using appropriate vocabulary. Be sure that students recognize how to identify the direction of horizontal translations. The function \( f(x + 2) = f(x - (-2)) \) translates the graph of \( f \) horizontally two unit to the left, not to the right.

See Connecting Transformation Definitions and Function Notation on page 279.

3. How does the graph of \( g(x) = \sqrt{x - 3} \) compare to the graph of \( f(x) = \sqrt{x} \)?
   It is a horizontal translation of the parent function 3 units to the right.

4. Sketch \( g \) and \( f \) from Item 3 on the same axes below.

5. What is the domain and range of \( g \)?
   The domain of \( g \) is \( x \geq 3 \). The range of \( g \) is \( y \geq 0 \).
   Multiple transformations can be applied to the basic function to create a new function.

6. Describe the transformations of \( f(x) = \sqrt{x} \) that result in the functions listed below.
   a. \( g(x) = -\sqrt{x} + 2 \)
      The graph of \( f \) reflected across the x-axis and translated 2 units to the left.
   b. \( h(x) = \sqrt{x - 3} + 4 \)
      The graph of \( f \) translated 3 units right and 4 units up.
7. Sketch the graph of each function in Item 6. Then state the domain and range for each function. Use a calculator to check your results.

8. Without graphing, determine the domain and range of 
   \[ f(x) = \sqrt{x + 5} - 1. \]
   Domain of \( f \): \( x \geq -5 \), Range of \( f \): \( y \geq -1 \).

Connecting Transformation Definitions and Function Notation

At this point, the definitions of various transformations should be very familiar to students. However, some students may still have difficulty recognizing some transformations with a new function. A visual cue like the one shown below may be helpful. Post this after students finish Item 8.

\[ f(x) = a \sqrt{x + b} + c \]

- vertical stretch by a factor of \( a \)
- horizontal translation of \( -b \)
- vertical translation of \( +c \)
ACTIVITY 5.2 Continued

5-10 Work Backward  Reinforce the meaning of points on the graph of the hull speed function (length at waterline, hull speed). Students should locate the point where \( x = 24 \) on the graph to find the speed of a boat with a 24-ft length at the waterline. Students should locate the point where \( y = 6 \) on the graph to find the length at the waterline of a boat with a hull speed of 6 knots.

11 Create Representations, Think/Pair/Share, Debriefing  Students will solve this equation at the end of Day 2 of this activity.

Suggested Assignment
CHECK YOUR UNDERSTANDING  p. 282, #1–6
UNIT 5 PRACTICE  p. 325, #6–9

Paragraph and Steps  
Note Taking, Interactive Word Wall, Vocabulary Organizer

The graph of the hull speed of a sailboat \( H \) is shown below.

<table>
<thead>
<tr>
<th>Hull Speed (knots)</th>
<th>Length at Waterline (ft)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
</tr>
<tr>
<td>6</td>
<td>4</td>
</tr>
<tr>
<td>7</td>
<td>5</td>
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<td>28</td>
<td>26</td>
</tr>
<tr>
<td>29</td>
<td>27</td>
</tr>
<tr>
<td>30</td>
<td>28</td>
</tr>
</tbody>
</table>

9. Use the graph to estimate the hull speed of a sailboat that is 24 ft long at the waterline. Hull speed is about 6.5 knots.

10. Use the graph to estimate the length at the waterline of a sailboat whose hull speed is 6 knots. The boat would be about 20 feet at the waterline.

11. Write an equation that could be used to determine the length at the waterline of a sailboat with a hull speed of 6 knots.

\[ 1.34/\sqrt{x} = 6 \]

To solve square root equations, follow these steps.

Step 1: Isolate the radical term.
Step 2: Square both sides of the equation.
Step 3: Solve for the unknown(s).
Step 4: Check for extraneous solutions.
EXAMPLE 1
Solve the equation \( \sqrt{x - 3} + 4 = 9. \)

Step 1: Isolate the radical.

\[ \sqrt{x - 3} = 5 \]

Step 2: Square both sides.

\[ (\sqrt{x - 3})^2 = (5)^2 \]

Step 3: Solve the equation.

\[ x - 3 = 25 \]

\[ x = 28 \]

Check the solution.

Step 4: Substitute 28 into the original equation.

\[ \sqrt{28 - 3 + 4} + 9 = 5 + 4 = 9 \]

EXAMPLE 2
Solve the equation \( x = \sqrt{x + 1} + 5. \)

Step 1: Isolate the radical.

\[ x = \sqrt{x + 1} + 5 \]

\[ x - 5 = \sqrt{x + 1} \]

Step 2: Square both sides.

\[ (x - 5)^2 = (\sqrt{x + 1})^2 \]

\[ x^2 - 10x + 25 = x + 1 \]

Step 3: Solve for \( x. \)

Possible solutions

\[ x = 3, 8 \]

Check the possible solutions.

Step 4: Try these A

Only \( x = 8 \) is a solution.

TRY THESE A
Solve each equation.

a. \( 2 - \sqrt{x + 1} = -5 \)

\[ x = 48 \]

b. \( \sqrt{x + 4} = x - 8 \)

\[ x = 12, x = 5 \] is an extraneous solution.

MINI-LESSON: Square Root Equations with More Than One Radical

For students who want to extend their learning, illustrate how to solve equations with more than one square root. These equations will come up again in Unit 7 when deriving equations for conics.

Solve: \( \sqrt{x - 3} = 4 - \sqrt{x + 5} \)

\[ (\sqrt{x - 3})^2 = (4 - \sqrt{x + 5})^2 \]

\[ x - 3 = 16 - 8\sqrt{x + 5} + x + 5 \]

\[ 3 = \sqrt{x + 5} \]

\[ x = 4 \]
ACTIVITY 5.2 Continued

**Marking the Text, Predict and Confirm, Create Representations, Quickwrite, Debriefing** This brief application problem will assess students’ ability to read the problem closely. Be sure that students use 24 ft in their equation, the length at the waterline, not 27 ft. This Item also provides an opportunity to assess communication skills. Some students may also wish to solve the equation graphically.

**Suggested Assignment**

CHECK YOUR UNDERSTANDING  
Suggested Assignment  
CHECK YOUR UNDERSTANDING  
p. 282, #7–11  
UNIT 5 PRACTICE  
p. 325, #10–12

**CHECK YOUR UNDERSTANDING**

1. Vertical stretch by a factor of 2, Horizontal translation 3 units to the right. Domain: \( x \geq 3 \), Range: \( y \geq 0 \).
2. Reflection across \( x \)-axis, Vertical translation 4 units up, Horizontal translation 1 unit to the left. Domain: \( x \geq -1 \). Range: \( y \leq 4 \).
3. Vertical stretch by a factor of 3, Horizontal translation of 5 to the right, vertical translation of 1 down. Domain: \( x \geq 5 \). Range: \( y \geq -1 \).
4. Parent function shown with dotted line.

5. and 6. See right side.

7. \( x = 17 \)

8. \( x = 5 \), \( x = 10 \) is an extraneous solution.

9. \( x = 100 \)

10. no solution

11. Answers will vary.

**SUGGESTED LEARNING STRATEGIES:** Marking the Text, Predict and Confirm, Create Representations, Quickwrite

12. Solve the hull speed equation you wrote in Item 11.  
   \[ x = 20.05 \text{ feet} \]

13. Maggie claims that her 27-foot sailboat My Hero has a hull speed of 7 knots. The length of her boat at the waterline is 24 ft. Is this claim reasonable? Explain why or why not.

Solve the equation \( 1.24\sqrt{x} = 7 \). The solution is \( x = 27.3 \text{ feet} \). The length at the waterline is only 24 ft so this is not a reasonable claim.

**CHECK YOUR UNDERSTANDING**

Write your answers on notebook paper or grid paper. Show your work.
Describe each function as a transformation of \( f(x) = \sqrt{x} \). State the domain and range.

1. \( f(x) = 2\sqrt{x} - 3 \)
2. \( f(x) = 4 - \sqrt{x} + 1 \)
3. \( f(x) = 3\sqrt{x} - 5 - 1 \)

Graph each function, using your knowledge of transformations.

4. \( f(x) = \sqrt{x + 1} - 3 \)
5. \( f(x) = -3\sqrt{x} + 1 \)
6. \( f(x) = 1 - \sqrt{x - 2} \)

**Solve for \( x \).**

7. \( \sqrt{x - 1} = 4 \)
8. \( x + \sqrt{x - 1} = 7 \)
9. \( 2 + \sqrt{x} = 12 \)
10. \( \sqrt{x + 4} + 7 = 3 \)

11. **MATHEMATICAL REFLECTION** What have you learned about graphing radical functions and solving radical equations in this activity?
Rational Exponents and Radical Expressions

A Mighty Wind

SUGGESTED LEARNING STRATEGIES: Shared Reading, Summarize/Paraphrase/Retell, Marking the Text, Note Taking, Think/Pair/Share

In 1805, Sir Francis Beaufort, a British admiral, devised a 13-point scale for measuring wind force based on how a ship’s sails move in the wind. The scale is widely used by sailors even today to interpret the weather at sea and has also been adapted for use on land. It was not until the early 1900s that the Beaufort scale was related to an average wind velocity in miles per hour using the equation \( v = 1.87B^2 \) where \( B \) is the Beaufort scale number.

1. The weather service issued a small-craft advisory with force 6 winds. Determine the wind speed predicted by the equation above, using a calculator. The wind speed for force 6 wind is approximately 27.5 mph.

The definition below relates a **rational exponent** to a radical expression. You can use this definition to evaluate expressions without a calculator.

**Definition of Rational Exponents**

\[
\begin{align*}
\sqrt[n]{a} &= a^{\frac{1}{n}} \text{ for } a > 0 \text{ and integer } n \\
\sqrt[n]{a^m} &= (\sqrt[n]{a})^m = a^{m/n} \text{ for } a > 0 \text{ and integers } m \text{ and } n
\end{align*}
\]

2. Without using a calculator, find the wind velocity predicted by the equation \( v = 1.87B^2 \) for gale force winds (\( B = 9 \)). Verify your answer, using a calculator. The wind speed is 50.49 mph.

The definition of rational exponents is also useful for simplifying expressions.

**EXAMPLE 1**

Simplify each expression, using the definition of rational exponents.

a. \( 81^{\frac{1}{2}} = \sqrt{81} = 9 \)

b. \( 9^{\frac{1}{3}} = \sqrt[3]{9} = 3 \approx 2.7 \)

c. \( 16^{\frac{1}{2}} = \sqrt{16} = 4 \)

TRY THESE A

Simplify each expression, using the definition of rational exponents. Write your answers on a separate sheet of paper. Show your work.

a. \( 81^{\frac{1}{2}} \cdot 81^{\frac{1}{2}} = \sqrt{81} \cdot \sqrt{81} = 9 \cdot 9 = 81 \)

b. \( 27^{\frac{1}{3}} = \sqrt[3]{27} = 3 \)

c. \( 1000^{\frac{1}{3}} = \sqrt[3]{1000} = 10 \)

**Math Tip**

\( \sqrt[n]{a} \) is called the \( n \)-th root of the radicand \( a \). For square roots, the index \( n \) is not written. The positive square root of 49 is \( \sqrt{49} = 7 \). The cube root of 8 is \( \sqrt[3]{8} = 2 \).

**ACADEMIC VOCABULARY**

- **rational exponent**

**Activity Focus**

- Rational and real exponents
- Properties of exponents
- Simplifying radical and rational exponent expressions
- Power functions \( f(x) = x^n \), where \( n \) is a rational number
- Solving equations and inequalities involving power functions with rational exponents

**Materials**

- Graphing calculator

**Chunking the Activity**

- \#1–2 Example 5–10
- Try These A \#7–8
- Try These B \#9–11
- Try These C \#12
- Example 3–5 \#13–14
- Try These D \#15
- Example 4–5 \#16
- Try These E

**First Paragraph**

Shared Reading, Summarize/Paraphrase/Retell, Marking the Text

**Paragraph and Definition box**

Note Taking

**2** Think/Pair/Share, Debriefing

TRY THESE A Think/Pair/Share, Debriefing

**Connection to Meteorology and the Beaufort Scale**

Search the internet for Beaufort Wind Scale and additional resources about the classification and measurement of sea conditions and wind speed.

<table>
<thead>
<tr>
<th>Force</th>
<th>Wind (knots)</th>
<th>Description</th>
<th>Force</th>
<th>Wind (knots)</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>0–1</td>
<td>0–3</td>
<td>calm, light air</td>
<td>7</td>
<td>28–33</td>
<td>near gale</td>
</tr>
<tr>
<td>2</td>
<td>4–6</td>
<td>light breeze</td>
<td>8</td>
<td>34–40</td>
<td>gale</td>
</tr>
<tr>
<td>3</td>
<td>7–10</td>
<td>gentle breeze</td>
<td>9</td>
<td>41–47</td>
<td>strong gale</td>
</tr>
<tr>
<td>4</td>
<td>11–16</td>
<td>moderate breeze</td>
<td>10</td>
<td>48–55</td>
<td>storm</td>
</tr>
<tr>
<td>5</td>
<td>17–21</td>
<td>fresh breeze</td>
<td>11</td>
<td>56–63</td>
<td>violent storm</td>
</tr>
<tr>
<td>6</td>
<td>22–27</td>
<td>strong breeze</td>
<td>12</td>
<td>64+</td>
<td>hurricane</td>
</tr>
</tbody>
</table>
EXAMPLE 2  Note Taking, Discussion Group Students extend the laws of exponents to rational and real numbers. Students will not be able to expand expressions any longer to solve problems. They will need to add, subtract, or multiply exponents to obtain the solutions.

TRY THESE B Think/Pair/Share, Simplify the Problem, Debriefing

EXAMPLE 3  Note Taking, Discussion Group Students are asked to simplify variable expressions. All variables are assumed to be positive.

TRY THESE C Think/Pair/Share, Simplify the Problem, Group Presentation, Debriefing

You may recall that the square root \( \sqrt{x} \) is treated as a function with domain \( x \geq 0 \). It is in this sense that \( \sqrt{x^2} = |x| \) for all values of \( x \). Notice, for example, \( \sqrt{(-3)^2} = \sqrt{9} = 3 = |−3| \). Sometimes the square root function is confused with the inverse relation to the quadratic function \( y = x^2 \). This inverse relation is \( x = y^2 \) and has two “branches,” a positive branch called the principal square root, and a negative branch.

**Suggested Assignment**
CHECK YOUR UNDERSTANDING p. 290, #1–6
UNIT 5 PRACTICE p. 325, #13–15

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**ACTIVITY 5.3 Continued**

**Math Tip**
The rules of exponents given \( a > 0, b > 0, \) and real numbers \( m \) and \( n \) are listed below.

\[
\begin{align*}
\alpha^n \cdot \alpha^m &= \alpha^{n+m} \\
\alpha^n &= \alpha^{-n} \\
(\alpha^n)^m &= \alpha^{nm} \\
\frac{1}{\alpha^n} &= \alpha^{-n} \\
\alpha^n \cdot \beta^n &= (\alpha \cdot \beta)^n \\
\alpha^{-n} &= \left(\frac{1}{\alpha^n}\right)^{-1} \\
\end{align*}
\]

**EXAMPLE 2**

Simplify each expression, using the properties of exponents.

\[
\begin{align*}
a. \quad 4^2 \cdot 4 &= 4^1 = 4^1 = 64 \\
b. \quad (3)^3 &= \sqrt[3]{3} = \sqrt[3]{3} = 3\sqrt[3]{3} \\
c. \quad \frac{\sqrt{2}}{\sqrt{3}} &= \frac{\sqrt{2}}{\sqrt{3}} = \sqrt[4]{2} = 2 \\
d. \quad (2)^{18} &= 2^{18} = 2^{18} = 77.7
\end{align*}
\]

**TRY THESE B**

Simplify each expression using the properties of exponents. Write your answers in the My Notes space. Show your work.

\[
\begin{align*}
a. \quad (9)^{\frac{1}{2}} &= 9^{\frac{1}{2}} = 3 = 27 \\
b. \quad 18^\frac{3}{2} \cdot 18 &= \sqrt[2]{18^3} = \sqrt[2]{18^3} = 18^3 \cdot 9 \cdot 2 = 18^3 \cdot 9 \cdot 2 = 972\sqrt{2} \\
c. \quad 3^{\frac{5}{2}} \cdot 2^{\frac{3}{2}} &= 6^{\frac{5}{2}} = 12.6
\end{align*}
\]

Variable expressions can also be simplified by using the properties of exponents.

**EXAMPLE 3**

Simplify \( x^\frac{1}{3} \cdot y^\frac{1}{2} \) and write in radical form. Assume \( x > 0 \) and \( y > 0 \).

**Step 1:** Write using definition of rational exponents. \( \sqrt{x^3} \cdot \sqrt{y^2} \)

**Step 2:** Multiply like radicals. \( \sqrt{x^3} \cdot \sqrt{y^2} \)

**Step 3:** Factor perfect squares. \( \sqrt{x^3} \cdot \sqrt{y} \)

**Step 4:** Simplify perfect squares. \( x \cdot \sqrt{y} \)

**TRY THESE C**

Simplify each expression and write in radical form. Assume all variables are greater than 0.

\[
\begin{align*}
a. \quad \frac{x^{\frac{1}{3}} \cdot y^{\frac{1}{2}}}{x^\frac{1}{3}} &= x^{\frac{1}{3}} \cdot y^{\frac{1}{2}} = xy^{\frac{1}{2}} \\
b. \quad \left(\frac{x^{\frac{1}{3}}}{}\right)^3 &= x^{\frac{1}{3}} \cdot x^1 = x^{\frac{4}{3}} \cdot \sqrt{x}
\end{align*}
\]
EXAMPLE 4

Solve the equation \( x^\frac{1}{3} = 36 \). Assume \( x > 0 \).

**Original equation**
\[ x^\frac{1}{3} = 36 \]

**Step 1:** Raise both sides to a power that makes the exponent on \( x \) equal to 1.
\[ (x^{\frac{1}{3}})^3 = 36^3 \]

**Step 2:** Simplify using the definition of rational exponents.
\[ x = (\sqrt[3]{36})^3 \]
\[ x = 6^1 \]
\[ x = 216 \]

TRY THESE D

Solve each equation. Assume \( x > 0 \).

a. \( x^\frac{1}{2} = 3 \)  
   \[ 81 \]

b. \( x^\frac{2}{3} = 32 \)  
   \[ 4 \]

c. \( 2x^3 = 54 \)  
   \[ 3 \]

3. If \( x \) could be any real number, not just a positive one, would there be any other solutions to the equation shown in Example 4 above? Answers will vary. Sample response: The number \(-216\) could also be a solution because after you take the cube root of \(-216\) you would square the number resulting in a positive 36.

4. Suppose the wind speed is measured at 30 mph.

a. Write an equation that could be solved to find the Beaufort number associated with this wind speed.
   \[ 1.87 \cdot B^2 = 30 \]

b. Solve the equation for \( B \).
   \[ B^2 = \frac{30}{1.87} \approx 16.042; \] \( B = 4 \) or \( B = 3.63 \). Round to the nearest scale value of 6.

c. Use the table on the first page of the activity to interpret the wind speed.
   The winds are strong enough that a small craft advisory would be issued.

**Math Tip**

Use the Power of a Power Property to isolate the variable so that it has an exponent of 1.

**Example:** \((x^\frac{1}{3})^3 = x\)

**Connect to AP**

In AP Calculus, you will work extensively with fractional and negative exponents.

**Activity 5.3 Continued**

**Example 4** Activating Prior Knowledge, Think/Pair/Share, Note Taking, Discussion Group, Quickwrite, Marking the Text, Summarize/Paraphrase/Retell, Create Representations, Look for a Pattern

Students will work extensively with fractional exponents in AP Calculus. They need to rewrite radical expressions so they have fractional exponents. They will also frequently rewrite expressions to contain negative exponents as well. Both of these rewrites are needed to find the derivative of certain functions.

For example the function \( f(x) = \frac{1}{\sqrt{x - 2}} \) would be re-written as \( f(x) = (x - 2)^{-\frac{1}{2}} \) prior to finding the derivative.
ACTIVITY 5.3 Continued

First Paragraph Activating Prior Knowledge, Note Taking

Look for a Pattern, Create Representations Make sure that students understand that the square root $\sqrt{x}$ is defined only for $x \geq 0$, and by definition of the meaning of square root, it follows that $\sqrt{x} \cdot \sqrt{x} = x$.

TRY THESE E Think/Pair/Share, Debriefing Carefully debrief these problems to make sure students are not making careless errors. The Math Tip is there to remind students about special binomial products used in parts b and c.

Paragraph and Explanation Box Note Taking, Interactive Word Wall, Vocabulary Organizer

EXAMPLE 5 Note Taking,

Look for a Pattern, Discussion Group, Debriefing This Item is intended to stimulate student dialogue regarding the process of rationalizing. Rather that think of rationalizing as a procedure that applies to radicals, students should see that multiplying by 1 in a variety of forms is a useful technique for simplifying a wide variety of expressions.

Math Tip

Difference of two squares

$$(a + b)(a - b) = a^2 - b^2$$

Perfect square trinomials

$$(a + b)^2 = a^2 + 2ab + b^2$$

TRY THESE

Simplify each expression.

a. $(\sqrt{5} + 3)(\sqrt{5} - 3)$

b. $(3 + \sqrt{3})^2$

c. $(\sqrt{5} + 1)(\sqrt{3} - 1)$

Radical expressions are often written in simplest form.

A radical expression with nth roots is in simplest form when:

- the radicand contains no perfect nth power factors other than 1
- there are no fractions in the radicand
- there are no radicals in the denominator of a fraction

The denominator of any radical expression can be rationalized by multiplying by the number 1.

Math Tip

The binomial expressions $a + b$ and $a - b$ are conjugates of each other.

EXAMPLE 5

Simplify each expression.

a. $\frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{\sqrt{5} \cdot \sqrt{5}} = \frac{2\sqrt{5}}{5}$

b. $\frac{\sqrt{8}}{\sqrt{5}} = \frac{2\sqrt{2}}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{2\sqrt{10}}{5}$

c. $\frac{3}{1 + \sqrt{5}} = \frac{3(1 - \sqrt{5})}{(1 + \sqrt{5})(1 - \sqrt{5})} = \frac{3 - 3\sqrt{5}}{1 - 5} = \frac{3 - 3\sqrt{5}}{-4}$
TRY THESE F

Simplify each expression. Write your answers in the My Notes space.

Show your work.

a. \( \frac{5}{\sqrt{10}} \)

b. \( \frac{1 + \sqrt{3}}{\sqrt{3}} \)

c. \( \frac{4}{\sqrt{5} - 2} \)

d. \( \frac{2 + \sqrt{3}}{3 + \sqrt{3}} \)

In a previous activity, you explored the graphs of square root functions.
The square root function is an example of a power function. A function of the form \( f(x) = x^n \) for any real number \( n \) is a power function.

7. Write the parent square root function as a power function.
   \( f(x) = x^{\frac{1}{2}} \)

8. List some other power functions that you have studied previously.
   Answers will vary. Sample response: \( f(x) = x \), \( f(x) = x^2 \), \( f(x) = x^3 \).

Other power functions with rational exponents have some interesting properties.

9. Graph the parent square root function and the power functions \( f(x) = x^3 \) and \( f(x) = x^1 \) on a graphing calculator. Then sketch the result on the grid below.

10. How do the graphs of these functions compare?
   Answers will vary. Sample response: In the first quadrant, the functions \( f(x) = x^1 \) and \( f(x) = x^3 \) are similar to the parent function \( f(x) = x^\frac{1}{2} \) but shrunk vertically. They all pass through \((0, 0)\) and \((1, 1)\). The domain of the function \( f(x) = x^3 \) is all real numbers, since it is reflected through the origin.

Create Representations Student sketches should label the intersection points \((0, 0)\) and \((1, 1)\), and each graph should be labeled clearly.

Summarize/Paraphrase/Retell, Look for a Pattern, Quickwrite Emphasize proper vocabulary on this Item.
11. Why is the function \( f(x) = x^{\frac{1}{3}} \) defined for all real values of \( x \) while the other two functions in Item 9 are not? The cube root of a negative number is a real number. The \( n \)'th root of a number when \( n \) is even is not a real number.

12. Without graphing, predict the domain of each function.
   
   a. \( f(x) = x^{\frac{1}{5}} \)
      all real numbers
   
   b. \( f(x) = x^{\frac{1}{6}} \)
      \( x \geq 0 \)

13. Graph the parent square root function and the power functions \( f(x) = x^{\frac{1}{3}}, f(x) = x^{\frac{1}{5}}, \) and \( f(x) = x^{\frac{1}{2}} \) on a graphing calculator. Then sketch the result on the grid below.

14. How do the graphs of these functions compare?
   
   Answers will vary. Sample response. They all contain the points \((0, 0)\) and \((1, 1)\). The graphs with powers less than one are “curved down” for \( x > 0 \). The functions with exponents with odd denominators are defined for all real numbers. The functions with exponents with even denominators are defined for \( x \geq 0 \).
**ACTIVITY 5.3 Continued**

15. How do the graphs of the functions in Item 13 help to explain the fact that an equation like $x^{\frac{3}{2}} = 8$ will only have 1 solution but an equation like $x^3 = 8$ will have 2 solutions? Answers will vary. Sample response: If you draw a horizontal line through the graph for some number greater than 0, it will intersect $f(x) = x^{\frac{3}{2}}$ twice but it will only intersect $f(x) = x^3$ once. This means there will be two $x$-values that satisfy the one equation and only one $x$-value that satisfies the second one.

16. Solve each equation and inequality, using a graphing calculator. Enter the left side as one function and the right side as another function. Solve graphically or algebraically.

   a. $2\sqrt{x} + 4 = 6$

   ![Graph of equation]

   The solution is $x = 5$.

   b. $x^\frac{3}{2} = 2$

   ![Graph of equation]

   The solutions are approximately -2.83 and 2.83.

**ACTIVITY 5.3**

Create Representations, Look for a Pattern, Discussion Group, Quickwrite, Debriefing

This Item uses the graph of the function to help students understand why there are two solutions to certain equations of the form $x^n = k$ where $n$ is a rational number, while others have one (or possibly no) solutions.

Marking the Text, Create Representations, Activating Prior Knowledge, Note Taking, Group Presentation, Debriefing

If students are not familiar with using a graphing calculator to solve an equation or inequality, provide some extra assistance on these problems.

Take time to explore the Math Tip with your class and have students take notes if solving inequalities graphically is unfamiliar to them. Ask them to recall how they did this in previous units.

If students are familiar with using technology in this way, consider assigning each group a different problem and then use group presentations to debrief student work.
Suggested Assignment
CHECK YOUR UNDERSTANDING
p. 290, #12–14
UNIT 5 PRACTICE
p. 326, #20–21

CHECK YOUR UNDERSTANDING

1. \(25\)
2. \(8\sqrt{8}\)
3. \(\frac{5\sqrt{3}}{9}\)
4. \(x^4y/x^3y\)
5. \(xy\sqrt{x}\)
6. \(\frac{x\sqrt{x}}{y}\)
7. \(x = 1000\)
8. \(x = 3\)
9. \(x - 16\sqrt{x} + 64\)
10. \(\sqrt{x^2 - y^2}\)
11. \(\sqrt{\frac{4}{2}} + 1\)
12. \(g\) is the graph of \(f\) stretched vertically by a factor of 2 and translated down 3 units.

13a. \(x > 47\)
b. \(x = 8\) or \(-8\)

Math Tip
If \(f(x) > k\), the solution interval will be those \(x\)-values in the function domain where \(f(x)\) is above the line \(y = k\).
If \(f(x) < k\), the solution interval will be those \(x\)-values in the function domain where \(f(x)\) is below the line \(y = k\).

CHECK YOUR UNDERSTANDING

Write your answers on notebook paper or grid paper. Show your work.

Solve each equation. Assume \(x > 0\).
7. \(5x^\frac{1}{3} = 50\)
8. \(4x^4 - 324 = 0\)

Simplify each expression.
1. \(\sqrt{125}\)
2. \(\sqrt{8} \cdot \sqrt{8}\)
3. \(\frac{(2\sqrt{3})^3}{2\sqrt{3}}\)
4. \(\sqrt{x^3} \cdot \sqrt{y^3}\)
5. \(\sqrt{\sqrt{x^3} \cdot \sqrt{y^3}}\)

Simplify each expression.
9. \(\sqrt{x - 8}^2\)
10. \(\sqrt{x + y} \cdot \sqrt{x - y}\)

Describe \(g(x) = 2x^\frac{1}{3} - 3\) as a transformation of \(f(x) = x^\frac{1}{3}\). Use this information to sketch the graph of \(g\).

Solve graphically or algebraically.
a. \(\sqrt{x + 3} - 3 > 4\)
b. \(3x^\frac{1}{3} = 12\)

How does the mathematics of this activity relate to what you have learned previously about simplifying expressions and solving equations?
Square Root Expressions, Equations, and Functions

A MIGHTIER WIND

1. The graph of a function \( g \) is shown below.

![Graph of function \( g \)](image)

a. Describe the graph as a transformation of \( f(x) = \sqrt{x} \).

b. Write the equation for \( g \).

c. State the domain and range of \( g \).

d. Find the inverse of \( g \). Be sure to include any restrictions on the domain of the inverse.

e. Use the graph or a table to solve the inequality \( g(x) > 7 \).

2. Solve the equation \( x + \sqrt{x} = 6 \).

3. Classify each statement as sometimes, always, or never true.

   a. \( 2 \cdot 4 = 8 \cdot 2 \)
   
   b. If \( x > 0 \), then \( \sqrt{x} \cdot \sqrt{2} = x \cdot 2 \).
   
   c. If \( x > 0 \), then \( \sqrt{x} + \sqrt{2} = x + 2 \).
   
   d. The only solution to \( x^2 = 4 \) is \( x = 8 \).

**Answer Key**

1a. It is the parent function vertically stretched by a factor of 2 and translated 3 units to the right.

b. \( g(x) = 2\sqrt{x} - 3 \)

c. Domain of \( g \): \( x \geq 3 \), Range of \( g \): \( y \geq 0 \)

d. \( g^{-1}(x) = \left( \frac{x}{2} \right)^2 + 3 \) for \( x \geq 0 \).

2. 4

This equation has an extraneous solution that students should eliminate upon checking their solution.

\[
\begin{align*}
x + \sqrt{x} &= 6 \\
\sqrt{x} &= 6 - x \\
(\sqrt{x})^2 &= (6 - x)^2 \\
x &= 36 - 12x + x^2 \\
x^2 - 13x + 36 &= 0 \\
(x - 4)(x - 9) &= 0 \\
x &= 4, 9
\end{align*}
\]

check: \( 4 + \sqrt{4} = 6 \)
check: \( 9 + \sqrt{9} \neq 6 \)

3. Students may find these Items difficult, but their responses will inform their thinking. For example, a student who answers always to part (c) does not have a complete understanding of solving rational equations. Revisiting the second half of Activity 5.3 will help those students.

3a. always  
   
   b. always  
   
   c. never  
   
   d. never
A MIGHTIER WIND

4. The International Tornado Intensity Scale (T-Scale) is widely used in Europe to describe the wind force of tornados. It was developed to extend the Beaufort scale to classify meteorological events with very high wind speeds. A TORRO force $T = 0$ is the same as the Beaufort force $B = 8$ and a TORRO force $T = 2$ is the same as the Beaufort force $B = 12$.

a. Write $T$ as a linear function of $B$, using the ordered pairs $(8, 0)$ and $(12, 2)$.

b. Use what you have learned about inverse functions to solve for $B$ as a function of $T$.

c. What is the value and meaning of $T(8)$?

d. Use composition of functions and the equation for wind velocity $v = 1.87 B$ to express $v$ as a function of $T$.

<table>
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<tr>
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### ACTIVITY 5.4 Investigative

**Introduction to Rational Functions**

**Activity Focus**
- Rational functions
- Domain and range
- Asymptotes
- Writing rational functions to model situations
- Graphing rational functions

**Materials**
- Graphing calculator

#### Chunking the Activity

#1–2 #8–9 #17  
#3 #10–13 #18  
#4–7 #14–16 #19–20

---

**My Notes**

The finance committee of a nonprofit summer camp for children is setting the cost for a 5-day camp. The fixed cost for the camp is $2400 per day, and includes things such as rent, salaries, insurance, and equipment. An outside food services company will provide meals at a cost of $3 per camper, per meal. Campers will eat 3 meals a day.

As a nonprofit camp, the camp must cover its costs, but not make any profit. The committee must come up with a proposal for setting the fee for each camper, based on the number of campers who are expected to attend each week.

1. Initially, the committee decides to calculate camper fees based on the fixed cost of the camp alone, without meals for the campers.
   - a. What is the total fixed cost for the five days? $12,000

   b. Complete the table below to determine the fee per camper that will guarantee the camp does not lose money.

<table>
<thead>
<tr>
<th>Number of Campers</th>
<th>Fee per Camper</th>
</tr>
</thead>
<tbody>
<tr>
<td>25</td>
<td>$480</td>
</tr>
<tr>
<td>50</td>
<td>$240</td>
</tr>
<tr>
<td>75</td>
<td>$160</td>
</tr>
<tr>
<td>100</td>
<td>$120</td>
</tr>
<tr>
<td>200</td>
<td>$60</td>
</tr>
<tr>
<td>500</td>
<td>$24</td>
</tr>
<tr>
<td>1000</td>
<td>$12</td>
</tr>
<tr>
<td>x</td>
<td>$12,000 (x)</td>
</tr>
</tbody>
</table>

**Math Tip**

Use the patterns you observe in the table to write an algebraic expression in the last row when there are \(x\) campers.
1c–d Create Representations

After students have completed their graphs, start a discussion about the domain and range of this function, emphasizing that the domain must be the positive integers. You might ask a question about the range such as, “What happens to the cost as the number of campers increases?”

2 Quickwrite, Debriefing

Be sure to emphasize the use of appropriate vocabulary here. Take time to refer students to their math notebooks for terms they should already know.

Points for discussion during a debriefing should include recognizing that the graph is discrete, decreasing, and does not have any intercepts. Students may realize that the graph does not extend below the x-axis, but the discussion of asymptotes can be deferred until later in this activity.

Math Tip:

You can use the values in your table to help you determine an appropriate scale for a graph.

3 Differentiating Instruction

If students have difficulty, post a vocabulary list for describing graphs, as shown below. Students may not be familiar with all the terms in the list, but should find the list helpful nonetheless.

Vocabulary for Describing Graphs

asymptotes increasing slope
continuous maximum point symmetry
decreasing minimum point x-intercepts
discontinuous range y-intercepts
domain rate of change zeros

Use a graphic organizer, interactive word wall, and vocabulary notebooks to help students review and organize these important terms.
ACTIVITY 5.4 Continued

b. Complete the table below to determine the fee per camper that will guarantee the camp does not lose money.

<table>
<thead>
<tr>
<th>Number of Campers</th>
<th>Fixed Cost plus the Cost of Meals</th>
<th>Fee per Camper</th>
</tr>
</thead>
<tbody>
<tr>
<td>25</td>
<td>$12,000 + 45(25) = $13,125</td>
<td>$525</td>
</tr>
<tr>
<td>50</td>
<td>$14,250</td>
<td>$285</td>
</tr>
<tr>
<td>75</td>
<td>$15,375</td>
<td>$205</td>
</tr>
<tr>
<td>100</td>
<td>$16,500</td>
<td>$165</td>
</tr>
<tr>
<td>200</td>
<td>$21,000</td>
<td>$105</td>
</tr>
<tr>
<td>500</td>
<td>$34,500</td>
<td>$69</td>
</tr>
<tr>
<td>1000</td>
<td>$57,000</td>
<td>$57</td>
</tr>
<tr>
<td>x</td>
<td>$12,000 + 45x</td>
<td>$12,000 + 45x</td>
</tr>
</tbody>
</table>

Using an appropriate scale, make a graph showing the relationship between the fee per camper, including meals, and the number of campers.

Write an algebraic rule for the fee per camper, including meals, as a function of the number of children in attendance.

\[ f(x) = \frac{12,000 + 45x}{x} \]

where \( x \) is the number of campers and \( f(x) \) is the fee per camper.

CONNECT TO AP

Describing the behavior of rational functions as they approach horizontal and vertical asymptotes provides an introduction to a more formal study of limits that will occur in calculus.

Rational functions are included in the list of prerequisites for AP Calculus. Students need to be able to analyze the behavior of rational functions and recognize their important features. When students describe the behavior of rational functions as the functions approach the horizontal and vertical asymptotes, they are building the conceptual foundation for a more formal study of limits that will occur in calculus.
ACTIVITY 5.4 Continued

5. Quickwrite

Students should recognize that as the number of children in attendance increases, the cost per camper approaches $45, and they may give $45 as a minimum cost. Others may suggest the minimum cost is $45 plus 12,000 divided by the maximum number of campers based on the camp’s capacity. When compared to their original response, the new minimum should reflect the $45 per camper cost of food.

6. Quickwrite, Self Revision/Peer Revision, Group Presentation, Debriefing

Suggested Assignment

CHECK YOUR UNDERSTANDING
p. 300, #1–2

UNIT 5 PRACTICE
p. 326, #22–24

7. a. Create Representations

As you circulate around the classroom, observe how students are completing the table. Guide them if needed to understand that they will be dividing by the number of paying children, not the total number of children in attendance.

CHECK YOUR UNDERSTANDING

1. The graph is increasing from $x = 1$, discrete, and non-negative.

2. domain: counting numbers; range: all real numbers greater than or equal to 120 and less than or equal to 200

4. 160

5. The model predicts 188 grizzly bears in the year 2018.

6a. vertical $x = 3$, horizontal $y = 1$

b. $x$-intercept is $-2$, $y$-intercept is $-\frac{2}{3}$.

c. See page 298.
8. (continued)

b. Using an appropriate scale, make a graph showing the relationship between the fee per paying camper and the number of campers.

<table>
<thead>
<tr>
<th>Number of Campers</th>
<th>Fee Per Paying Camper</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>200</td>
</tr>
<tr>
<td>100</td>
<td>250</td>
</tr>
<tr>
<td>150</td>
<td>300</td>
</tr>
<tr>
<td>200</td>
<td>350</td>
</tr>
</tbody>
</table>

8. (continued)

c. Write an algebraic model for the fee per paying camper as a function of the number of campers in attendance.

\[ f(x) = \frac{12,000 + 45x}{x - 30} \]

where \( x \) is the number of campers and \( f(x) \) is the fee per camper.

9. Based on your work so far, is there a minimum camper’s fee? If so, what is it? Explain.

The minimum would be 45 plus 13,350 (the fixed cost plus meals for 30 scholarship campers) divided by the number of paying campers.

10. How does your answer to Item 9 differ from the one you gave for Item 5?

It takes into account the fact that the paying campers have to absorb the cost on the scholarship campers meals and their share of the fixed costs.

11. How does your graph in Item 8(b) compare to the one in Item 4(c)?

The graphs are decreasing, discrete, and approaching 45, but the \( x \)-values must be greater than 30.

You can discuss rewriting the function \( f(x) = \frac{12,000 + 45x}{x - 30} \) to make the horizontal asymptote more apparent. Using long division, the function can be rewritten as \( f(x) = 45 + \frac{13,350}{x - 30} \).

In this situation, the number 13,350 represents the fixed cost of $12,000 plus the cost of meals for the 30 students on fee waiver, 30(45) or $1350. As the number of paying campers increases, the amount each one pays toward the $13,350 decreases, but each paying camper still pays $45 for meals.
DEBRIEFING Students should be able to evaluate the function they have written to determine the answer to this Item. Students should recognize that a value of –2625 for the cost per camper is not appropriate. There must be more than 30 campers with at least one paying, so the domain of the function is integers greater than 30.

WORK BACKWARD Students may solve the equation analytically or by finding the intersection of f and 80 on a graphing calculator. Make sure students subtract 30 from their solution to reflect the actual number of paying campers.

WORK BACKWARD, QUICKWRITE, DEBRIEFS These Items emphasize inverse functions. To find the number of campers as a function of the fee per paying camper, let y equal f(x), exchange y and x, and solve for y.

TEACHER TO TEACHER You can review the concept of the inverse of a function at this point by looking at the graphs of the functions with respect to the line y = x in a square viewing window on the calculator and by comparing the domain and range values. Pick an ordered pair (a, b) and have students demonstrate the following: If f(a) = b, then g(b) = a, and f(g(b)) = b and g(f(a)) = a.

PREWITING, RAFT This Item is an opportunity for students to communicate their understanding of the work they have done so far. You might assign this Item for homework.

SUGGESTED LEARNING STRATEGIES: Guess and Check, Work Backward, Quickwrite, Prewriting, RAFT

12. If the number of campers is 25, what is the fee per paying camper? What does your answer tell you about the limitations of this model? f(25) = –2625. This answer means that the function is not valid for 30 campers or less.

13. What is the domain of the function for the fee per paying camper? The counting numbers where x > 30.

14. Last year the weekly camper fee was $80. If the camp charges the same amount and grants 30 scholarships, what is the minimum number of paying campers that must attend so the camp does not lose money?

15. Express the number of campers as a function of the fee for each paying camper.

16. What is the relationship between the function in Item 8(c) and the function in Item 13? The two functions are inverses.

17. On a separate sheet of paper, write a proposal for setting the fee per camper. Be sure to include these items. Answers will vary.

• the proposed fee
• the minimum number of campers needed to break even
• the maximum possible income for the proposed fee
• mathematics to support your reasoning

SUGGESTED ASSIGNMENT
CHECK YOUR UNDERSTANDING
p. 300, #3-5

UNIT 5 PRACTICE
p. 326, #25-26
MINI-LESSON: Identifying Rational Functions

Struggling students may need additional help to recognize and understand what is and what is not a rational function. Give them these examples and have them work in groups to sort them into two groups: Rational Function and Not Rational Function.

1. $f(x) = \frac{x^2 + 2}{3}$  
2. $f(x) = x^2 + 3x + \frac{1}{2}$  
3. $f(x) = \frac{2}{x} + 3$  
4. $f(x) = x^2 + 3x + \frac{1}{2}$  
5. $f(x) = \frac{x^2}{3x + 2}$

3 and 5 are rational functions.
19. What are the vertical and horizontal asymptotes of
   \( f(x) = \frac{12000 + 45x^2}{x - 30} \)
   Vertical asymptote: \( x = 30 \); Horizontal asymptote: \( y = 45 \).

20. Sketch the graph of the function in Item 19 on the axes below.
   Indicate the scale, label the intercepts, and include the horizontal and vertical asymptotes.

SUGGESTED LEARNING STRATEGIES: Think/Pair/Share, Look for a Pattern, Create Representations, Group Presentation

---

**SUGGESTED LEARNING STRATEGIES:**

**Think/Pair/Share**

The horizontal asymptote can be found using the graphing calculator or by performing long division. A later activity will teach students to find horizontal asymptotes by comparing the degree of the numerator and denominator.

**Look for a Pattern, Create Representations, Group Presentation, Debriefing**

When debriefing this Item, focus on finding intercepts and asymptotes as key features to identify when sketching a rational function. Later activities will further hone student skills with graphing a rational function without the aid of a calculator. For now, students should use a calculator as needed and make the connections between what they are viewing on the calculator and what they can determine from the equation.

**Suggested Assignment**

CHECK YOUR UNDERSTANDING
p. 300, #6–8

UNIT 5 PRACTICE
p. 326, #27–30

---

**CHECK YOUR UNDERSTANDING**

1-6b. See p. 296.
   c. See p. 298.
   7a. vertical \( x = 5 \), horizontal \( y = 2 \)
   b. \( x \)- and \( y \)-intercepts are both 0.
   c. [Graph of a rational function showing vertical and horizontal asymptotes]

8. Answers may vary.

---

**TEACHER TO TEACHER**

It is possible for a function \( f \) to have two horizontal asymptotes: one corresponding to the value approached as \( x \) increases with bound; and one as \( x \) decreases without bound. The phrase *absolute value of \( x \)* in the description on the previous page suggests that only one horizontal asymptote is possible. As an example, the function \( f(x) = \frac{\sqrt{4x^2 + 5}}{x - 3} \) has two horizontal asymptotes.
The amount of dissolved oxygen in a body of water decreases as the water temperature increases. Dissolved oxygen needs to be at sufficient levels to sustain the life of aquatic organisms such as fish. The table shows the temperature $t$ and the corresponding amount of dissolved oxygen $D$ in a stream that flows into Lake Superior on several dates from May to August.

<table>
<thead>
<tr>
<th>Date</th>
<th>$t$ (Celsius)</th>
<th>$D$ (mg O(_2)/L)</th>
<th>Product $t \times D$</th>
</tr>
</thead>
<tbody>
<tr>
<td>May 1</td>
<td>11.5</td>
<td>10.6</td>
<td>121.9</td>
</tr>
<tr>
<td>May 15</td>
<td>12.5</td>
<td>9.8</td>
<td>122.5</td>
</tr>
<tr>
<td>June 1</td>
<td>13.0</td>
<td>9.5</td>
<td>123.5</td>
</tr>
<tr>
<td>June 15</td>
<td>14.0</td>
<td>8.7</td>
<td>121.8</td>
</tr>
<tr>
<td>July 1</td>
<td>14.5</td>
<td>8.5</td>
<td>123.25</td>
</tr>
<tr>
<td>July 15</td>
<td>15.0</td>
<td>8.1</td>
<td>121.5</td>
</tr>
<tr>
<td>Aug 1</td>
<td>16.5</td>
<td>7.4</td>
<td>122.1</td>
</tr>
</tbody>
</table>

1. Graph the data above as a set of points on the axes. Answer to Item 1 is set of points; curve is answer to Item 7.

2. Are these data linear? Explain why or why not. Although the data may appear linear from the graph, if you use the table, you can see that slope between pairs of points is not constant.

3. Add a fourth column to the table, showing the product of $t$ and $D$. See above.

4. What do you observe about the products of $t$ and $D$ that you recorded in the table? They are very close to each other—between 121.5 and 123.5.

Connect to Ecology

Many factors contribute to water quality, but without enough dissolved oxygen, all aquatic life forms will suffer. In reality, most aquatic life living in a stream or lake would migrate to a cooler or warmer portion of the body of water to meet biological needs. Water temperature varies as the depth increases.

From a theoretical standpoint, the water temperature and dissolved oxygen levels do vary inversely. However, the data in this activity do not exhibit a perfect inverse variation because many factors affect the daily water temperature of a stream over the course of a summer. In general, from early spring through early August, average water temperatures will rise in the geographic locations associated with these data.
ACTIVITY 5.5 Continued

Definition Box and First Paragraph Note Taking, Interactive Word Wall, Vocabulary Organizer, Summarize/Paraphrase/Retell

Take time to explain the definition of inverse variation. Discuss what it means to create a mathematical model. While a model may not fit the data perfectly, it is useful for making predictions and explaining the nature of real-world phenomena.

Think/Pair/Share When you debrief this Item, take note of how students decided on their value for \( k \). Some may average the products they found, while others may estimate it in other ways. At this point, accept all methods.

Create Representations Some students may have difficulty with using the definition given in terms of \( x \) and \( y \) if a question uses different variables.

Create Representations, Quickwrite If necessary, show students how to enter data into their calculators. Have students create two lists, one for the independent values \( (t) \) and one for the dependent values \( (D) \). Then show students how to use the statistics plotting features to create a scatter plot of the points \( (t, D) \). Encourage students to use the graph scale in Item 1 to set up their window. If students select any value for \( k \) near 122, their graphs will be very close to the graph for Item 1.

Work Backward, Debriefing There are many ways to solve this problem. Some students may try guess and check, others may use algebra or their calculator to graph the line \( y = 6 \) and find where it intersects their curve. Use whiteboards for the results of Items 7–9 and encourage multiple solution methods.

Inverse Variation Equation

When the product of two variable quantities \( x \) and \( y \) is constant, the two variables are said to vary inversely.

If \( xy = k \) and \( x \neq 0 \), then \( y = \frac{k}{x} \), where \( k \) is the constant of variation.

Although the products of \( t \) and \( D \) from Item 3 are not constant, the products are close in value. When you use mathematics to model a real-world situation, the functions do not always give exact results.

5. If you use inverse variation to model the dissolved oxygen and temperature relationship, what value would you choose for \( k \)?

Answers will vary. Sample answer: 122.5, which is the average of 121.5 and 123.5.

6. Write an inverse variation equation relating \( t \) and \( D \) that shows a constant product. Then solve the equation for \( D \).

\[
\begin{align*}
D &= \frac{122.5}{t} \\
D &= 122.5 \\
\end{align*}
\]

7. Use your calculator to make a scatter plot of the points \( (t, D) \) and graph the equation from Item 6 on the axes in Item 1.

See graph in Item 1 for answer to Item 7.

8. How well does the model that you created fit the data?

Answers will vary. Sample answer: It fits the points very well.

9. When dissolved oxygen is less than 6 mg O/L, salmon are in danger.

Use the model to find the maximum safe temperature for salmon.

Solve: \( \frac{122.5}{t} \geq 6 \), so \( t \leq 20.4 \). If the temperatures stay below 20.4 degrees Celsius, then the fish will be safe.

TECHNOLOGY Tip

If students are familiar with using the regression equation features of a calculator, they could easily find a power regression. Their equation should be close to \( D = 124t^{-1.04} \), which further confirms that an inverse variation model is appropriate. If students have not used the regression features of the calculator before, you could have them find both a linear and a power model. To better compare the two models, have them graph the two equations and then re-size their viewing window using the ZOOMSTAT. Students will clearly see that a power regression fits the data better than a linear regression when viewing the graph in a smaller window.

Ask these follow-up questions: What is the power of \( x \)? Does this confirm that an inverse variation model is appropriate?
TRY THESE A  Use an inverse variation equation to solve each problem.

a. $y$ varies inversely as $x$. When $x$ is 5, $y$ is 10. Find $y$ when $x$ is 18.
   
   \[ k = \frac{10 \cdot 5}{50} \]  
   \[ y = \frac{50}{x} \quad \text{when} \quad x = 18, \quad y = 2.78 \]

b. The length of a rectangle varies inversely as its width. If the area is 40 in.$^2$ and the width is 12.5 in., find the length of the rectangle.
   
   The length is 3.2 inches.

b. Boyle’s law says that the volume of a gas in a closed container at constant temperature is inversely proportional to the pressure of the gas. Suppose 5 L of a gas are at a pressure of 2.0 atmospheres. What will be the volume if the pressure is increased to 3.0 atmospheres?
   
   \[ k = 5 \cdot 2 = 10 \]
   \[ V = \frac{10}{p} \quad \text{when} \quad p = 3, \quad v = \frac{10}{3} \text{ L} \]

Another type of variation is direct variation. Two unknowns $x$ and $y$ vary directly if they are related by the equation $y = kx$ where $k$ is a nonzero constant. The graph of a direct variation equation is a line passing through the origin.

EXAMPLE 1

The area of a rectangle with a fixed width varies directly as its length. When the area is 40 cm$^2$, the length is 5 cm. Write a direct variation equation for the area of the rectangle. Use the equation to determine the area when the length is 20 cm.

\begin{align*}
\text{Step 1:} & \quad \text{Use the direct variation formula with } A \text{ as area}, \\
& \quad A = k \cdot l \\
& \quad \text{as length, } k \text{ as constant of proportionality.} \\
\text{Step 2:} & \quad \text{Substitute } A = 40 \text{ and } l = 5 \text{ to find } k. \\
& \quad 40 = 5k \\
& \quad k = 8 \\
\text{Step 3:} & \quad \text{Write the direct variation equation for this situation.} \\
& \quad A = 8l \\
\text{Step 4:} & \quad \text{Find } A \text{ when } l = 20. \\
& \quad A = 8(20) \\
& \quad A = 160 \\
\text{Solution:} & \quad \text{The area is 160 cm}^2.
\end{align*}

TRY THESE B  Use a direct variation equation to solve the problem.

a. $y$ varies directly as $x$. When $x = 3$, $y = 30$. Find $y$ when $x = 7$.
   
   \[ y = 10x \quad \text{when} \quad x = 7, \quad y = 70. \]

b. Distance traveled varies directly as time if the speed is constant. A 500-mi trip takes 8 h at a constant speed. How long would it take to travel 400 mi at the same speed?
   
   \[ d = 62.5t. \quad \text{When } d = 400 \text{ miles, } t = 6.4 \text{ hours}. \]

MINI-LESSON: Types of Variations

Variation problems become easier once students understand the vocabulary.

<table>
<thead>
<tr>
<th>Words</th>
<th>Basic Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$ varies directly as $x$</td>
<td>$y = kx$</td>
</tr>
<tr>
<td>$y$ varies jointly as $x$ and $z$</td>
<td>$y = kxz$</td>
</tr>
<tr>
<td>$y$ varies inversely as $x$</td>
<td>$y = \frac{k}{x}$</td>
</tr>
</tbody>
</table>

Combined variation includes both direct and inverse. For example, $V$ varies directly as $x$ and inversely as the square of $y$ in the equation $V = \frac{kA}{y^2}$. 

ACTIVITY 5.5  Continued

TRY THESE A  Create Representations, Marking the Text, Group Presentation, Vocabulary Organizer, Interactive Word Wall, Note Taking

Paragraph Vocabulary Organizer, Interactive Word Wall  Direct variation is introduced briefly in this activity. For students who are not familiar with direct variation from their Algebra 1 course, review this topic now. See also Mini-Lesson: Types of Variations below.

EXAMPLE 1  Note Taking

TRY THESE B  Create Representations, Marking the Text, Debriefing

Suggested Assignment

CHECK YOUR UNDERSTANDING

p. 308, #1–7

UNIT 5 PRACTICE

p. 326, #31–34
The focus of the rest of the activity is developing the parent function $f(x) = \frac{1}{x}$ and then using transformations to graph other rational functions. Remind students of their previous work with rational functions, including vocabulary, by reviewing the graph they sketched for Item 20 in Activity 5.4 and pointing out the terms that you have added to your interactive word wall.

**ACTIVITY 5.5 Continued**

10. **Vocabulary Organizer, Interactive Word Wall, Create Representations** Emphasize the need to graph key points of the function. Students should graph two points at a minimum, (1, 1) and (–1, –1). Until students become familiar with the graph, encourage at least 4 more points in their table and on their graph. They can also use the symmetry of the function to help them sketch the parent function quickly and accurately.

11. **Quickwrite, Group Presentation, Debriefing** Check how students are using the vocabulary introduced in earlier units. They should be able to describe this function and identify where it is positive and negative. Although students have already studied end behavior, rational functions have different end behavior than polynomials. Make sure students include the asymptotes in their descriptions as well.

**Paragraph Vocabulary Organizer, Interactive Word Wall**

12. **Create Representations** Students explore the effect of multiplication by a constant in this item and the next two items. Here students should first graph the functions on their graphing calculators to make the graphing easier. Then they should make a quick sketch on the axes provided.

---

The rational function $f(x) = \frac{1}{x}$ is an example of an inverse variation equation whose constant of variation is 1.

10. Make a table of values in the My Notes section. Graph the parent rational function $f(x) = \frac{1}{x}$ below.

11. Describe the key features of $f(x) = \frac{1}{x}$. Use appropriate mathematics vocabulary in your description.

Answers will vary. Sample answer: The function has a vertical asymptote at $x = 0$ and a horizontal asymptote at $y = 0$. The domain is all real numbers except for $x = 0$, and the range is all real numbers except for $y = 0$. When $x < 0$, the function is negative. When $x > 0$, the function is positive.

Functions like the one modeling dissolved oxygen and temperature are a vertical stretch of the parent graph $f(x) = \frac{1}{x}$.

12. Enter the functions $f(x) = \frac{1}{2}$, $g(x) = \frac{2}{x}$, and $h(x) = \frac{5}{x}$ into your graphing calculator. Sketch the graphs on the axes below.

---

There is a relationship between the natural logarithms and the rational function $f(x) = \frac{1}{x}$. The area under the function $f$ from 1 to any number $a > 0$ is equal to the $\ln a$. Students will actually define the natural logarithm function using a definite integral in calculus.
ACTIVITY 5.5 Continued

13. How do the y-values of \( g \) and \( h \) compare to those of the parent graph?
   Answers will vary. Sample answer: The y-values of \( g \) are 2 times the parent function’s y-values. The y-values of \( h \) are 5 times the parent function’s y-values.

14. Describe the similarities and the differences in the graphs of those three functions.
   All three graphs have the same vertical and horizontal asymptotes. As the coefficient gets larger, the graph is stretched farther away from the axes.

15. Sketch the parent graph \( f(x) = \frac{1}{x} \) and the graph of \( k(x) = \frac{3}{x} \) on the same axes without using your graphing calculator.

16. Without using your calculator, predict what the graph of \( f(x) = -\frac{1}{x} \) will look like. Confirm prediction by graphing both functions on your calculator.
   Answers will vary. Sample answer: The graph will be reflected across the x-axis.

Math Tip
Given \( y = f(x) \), the function \( y = -f(x) \) represents a vertical reflection of the original function whose y-values have been multiplied by \(-1\).
17. Sketch the graph of each function and then describe it as a transformation of the parent graph \( f(x) = \frac{1}{x} \). The first graph has been done for you.

**MATH TERMS**

Given \( y = f(x) \), the function \( y = f(x + c) \) results in a **horizontal translation** of the original function and \( y = f(x) + c \) results in a **vertical translation** of the original function.

**Transformation:**

- **The graph is translated 2 units to the left.**
- **The graph is translated 2 units to the right.**
- **The graph is translated up 2 units.**
- **The graph is translated down 2 units.**
TRY THESE C
Describe each function as a transformation of $f(x) = \frac{1}{x}$.

- **a.** $f(x) = \frac{1}{x} + 1$
  - The graph is translated 1 unit to the left.

- **b.** $f(x) = \frac{1}{x} - 3$
  - The graph is translated down 3 units.

- **c.** $f(x) = \frac{1}{x - 3} + 3$
  - The graph is translated 5 units to the right and up 3 units.

EXAMPLE 2
Describe the function $f(x) = \frac{-2}{x - 3} + 1$ as a transformation of $f(x) = \frac{1}{x}$.

Identify the $x$- and $y$-intercepts and the asymptotes. Sketch the graph.

<table>
<thead>
<tr>
<th>Transformations</th>
<th>Asymptotes</th>
</tr>
</thead>
<tbody>
<tr>
<td>• vertical stretch by a factor of 2</td>
<td>$x = 3$</td>
</tr>
<tr>
<td>• horizontal translation 3 units to the right</td>
<td>$y = 1$</td>
</tr>
<tr>
<td>• vertical translation 1 unit up</td>
<td>$x = 0$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Intercepts</th>
<th>Graph</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$-intercept: $f(0) = \frac{1}{3}$</td>
<td><img src="image" alt="Graph" /></td>
</tr>
<tr>
<td>$x$-intercept: Solve $f(x) = 0$.</td>
<td><img src="image" alt="Graph" /></td>
</tr>
<tr>
<td>$\frac{-2}{x - 3} + 1 = 0$</td>
<td><img src="image" alt="Graph" /></td>
</tr>
<tr>
<td>$\frac{2}{x - 3} = -1$</td>
<td><img src="image" alt="Graph" /></td>
</tr>
<tr>
<td>$2 = -1(x - 3)$</td>
<td><img src="image" alt="Graph" /></td>
</tr>
<tr>
<td>$x = 1$</td>
<td><img src="image" alt="Graph" /></td>
</tr>
</tbody>
</table>

TRY THESE D
On a separate sheet of grid paper, describe each function as a transformation of $f(x) = \frac{1}{x}$. Identify the $x$- and $y$-intercepts and the asymptotes. Sketch the graph.

- **a.** $f(x) = \frac{1}{x} + 1$

- **b.** $f(x) = \frac{1}{x} - 2$

- **c.** $f(x) = 3 + \frac{4}{x - 2}$

- **b.** Reflected about the $x$-axis, translated 1 unit left and 2 down.
  - Vertical Asymptote: $x = -1$,
  - Horizontal Asymptote: $y = -2$,
  - $x$-intercept $-1.5$,
  - $y$-intercept $-3$

- **c.** Vertically stretched by a factor of 4, translated 2 units right and 3 up.
  - Vertical Asymptote: $x = 2$,
  - Horizontal Asymptote: $y = 3$,
  - $x$-intercept $\frac{2}{3}$, $y$-intercept 1.
CHECK YOUR UNDERSTANDING

1. 20
2. Let \( t = \text{time and } r = \text{rate} \)
   \[ 1.5 \cdot 50 = 75 \]
   \[ t = \frac{75}{r} \]
3. The distance is 75 miles.
4. The trip will take about 1.15 h.
5. Yes, it does, because the products of \( x \) and \( y \) are constant, 48.
6. No, it does not. The data have a constant slope, so they are linear, but quotients of the ordered pairs are not constant, so it doesn’t fit the form \( y = kx \).
7. \( y = 0.25x \). When \( x = 5 \), \( y = 1.25 \).
8. The graph is stretched vertically by a factor of 2 and translated 5 units to the left.
9. The graph is reflected across the \( x \)-axis and translated 3 units to the right and up 5 units.
10. The graph is stretched by a factor of 10 and translated 4 units to the left and down 2 units.
11. and 12. See below.
13. Answers may vary.
Rational expressions can be simplified and combined, using the operations of addition, subtraction, multiplication and division.

Writing rational expressions in simpler forms and combining them helps you to understand and graph rational functions and solve equations.

To simplify a rational expression, factor the numerator and denominator. Identify the restrictions on the variable x that make the denominator in the expression equal to zero. Then, divide out the common factors.

**EXAMPLE 1**

Simplify each expression.

Original expression

| Step 1: Identify the restrictions on x. Set the denominators equal to zero. |
|-----------------------------|---------------------|
| A. \( \frac{x^2 + 5x - 14}{x^2 - 4} \) | B. \( \frac{2x^2 + 7x + 3}{x^2 + 7x + 12} \) |
| \( x^2 - 4 = 0 \) | \( x^2 + 7x + 12 = 0 \) |
| \((x + 2)(x - 2) = 0\) | \((x + 3)(x + 4) = 0\) |
| \(x + 2 = 0 \) or \(x - 2 = 0\) | \(x + 3 = 0 \) or \(x + 4 = 0\) |
| \(x = -2 \) or \(x = 2\) | \(x = -3 \) or \(x = -4\) |

**Step 2:** Factor the numerators and denominators.

- A. \( \frac{x^2 + 5x - 14}{x^2 - 4} = \frac{(x + 7)(x - 2)}{(x + 2)(x - 2)} = \frac{x + 7}{x + 2}, x \neq 2, -2 \)
- B. \( \frac{2x^2 + 7x + 3}{x^2 + 7x + 12} = \frac{(2x + 1)(x + 3)}{(x + 4)(x + 3)} = \frac{2x + 1}{x + 4}, x \neq -3, -4 \)

**Step 3:** Divide out common factors.

- A. \( \frac{x + 7}{x + 2}, x \neq 2, -2 \)
- B. \( \frac{2x + 1}{x + 4}, x \neq -3, -4 \)

**TRY THESE A**

Simplify. Identify any restrictions on x. Write your answers on notebook paper. Show your work.

a. \( \frac{x^2 + 20x + 36}{x^2 - 4x} \) b. \( \frac{2x^2 - 2x - 15}{2x^2 + 3x - 9} \) c. \( \frac{3x^2 - 9x}{3 - x} \)

\( x + 10 \) \( x(x - 2), x \neq 0, -2, 2 \)

**TRY THESE A**

To understand the restrictions on the variables in Examples 1–3, students should think of these rational expressions as functions. As functions, each has a domain. Students then determine the domain of each rational function (i.e., the restrictions on the variable represent values that are not part of the domain). Once the domain has been identified, common factors can be cancelled, provided that the new expression still preserves the domain that it previously had. In the context of functions, all this makes more sense, because two functions are equal if their domains are identical and their values are equal over their identical domains.

**Math Tip**

When a rational function has a denominator of zero, restrictions on the variable are needed.

**Math Tip**

Make sure that you use the original rational expression when identifying restrictions on the variable.

**EXAMPLE 1 Note Taking**

Model the example and check for understanding. Remind students to identify restricted values before they cancel any common factors.

**TRY THESE A**

Think/Pair/Share, Create Representations, Simplify the Problem, Debriefing
ACTIVITY 5.6 continued

When multiplying two rational expressions, some students fail to cancel common factors that appear in the numerator and denominator of the same expression. They may be confusing cancelling a common factor with cross-multiplying in proportions.

The multiplicative identity does not apply to the operation of addition. For example, $\frac{x + 2}{x + 3} \neq \frac{2}{3}$. Students must understand that they can cancel only factors in a product, not addends in a sum.

EXAMPLES 2–3  Note Taking

Identification of restricted values was specifically excluded from these problems because many restricted values occur, especially when dividing rational expressions. The denominator of the dividend cannot be zero and neither the numerator nor the denominator of the divisor can be zero.

Second Paragraph  Marking the Text

TRY THESE B  Think/Pair/Share, Create Representations, Group Presentation

It is important to have students identify all the restricted values throughout this activity. In Example 3 on this page, the variable cannot equal the following values: $-3$, $-2$, and $2$.

Suggested Assignment

CHECK YOUR UNDERSTANDING  p. 316, #1–5

UNIT 5 PRACTICE  p. 327, #40–42

MINI-LESSON: Opposites

Problem (c) in Try These A on the previous page required students to recognize that the expression $3 - x$ can be written as $-1(x + 3)$. The expressions $3 - x$ and $x - 3$ are additive inverses. When simplifying, the quotient of an expression and its additive inverse will always equal $-1$.

Each step in the solution of Problem (c) is shown below:

$$x^2 - 9x \quad \text{and} \quad x^2 - 5x - 50 \quad \text{are additive inverses.}$$

$$\frac{x^2 - 9x}{3 - x} = \frac{x(x - 3)(x + 3)}{-(3 + x)} = \frac{x(x + 3)}{-(3 + x)} = -x(x + 3)$$

$$\frac{x^2 - 5x - 50}{x - 10} \quad \frac{2x + 2}{x - 2x - 3}$$

To multiply rational expressions and express the product in lowest terms, factor the numerator and denominator of each expression. Then, divide out any common factors.

EXAMPLE 2

Multiply the expression. Assume no denominator is zero.

$$\frac{x^2 - 8}{x^2 - 1} \quad \frac{x^2 + 2x + 1}{x^2 - x - 2x}$$

Step 1:  Factor the numerators and denominators.

$$\frac{x(x + 1)}{(x + 1)(x - 1)} \quad \frac{(x + 1)(x + 1)}{(x + 1)(x - 2)}$$

Step 2:  Divide out common factors.

$$\frac{x}{x - 1} \quad \frac{x + 1}{x - 2}$$

To divide rational expressions, write division as multiplication and then finish simplifying the expression.

EXAMPLE 3

Divide the expression. Assume no denominator is zero.

$$\frac{x^2 + 5x + 6}{x^2 - 4} \quad \frac{x^2 - 3x - 4 + 5}{x^2 - 4}$$

Step 1:  Write as multiplication.

$$\frac{(x + 2)(x + 3)}{x^2 - 4} \quad \frac{(3x + 2)(x - 2)}{(x + 2)(x - 2)}$$

Step 2:  Factor the numerators and denominators.

$$\frac{(x + 1)(x - 1)(x + 3)}{(x + 2)(x - 2)}$$

Step 3:  Divide out common factors.

$$\frac{x + 1}{x - 2}$$

TRY THESE B

Perform the indicated operation. Assume no denominator is zero.

$$a. \frac{2x + 4}{x^2 - 25} \quad \frac{x^2 - 5x - 50}{4x^2 - 16} \quad \frac{x^2}{x + 3}$$

$$b. \frac{6x^2}{3x^2 - 27} \quad \frac{2x + 2}{x - 2x - 3}$$

WRITE YOUR ANSWERS IN YOUR NOTEBOOK.
SUGGESTED LEARNING STRATEGIES: Summarize/Paraphrase/Retell, Note Taking, Graphic Organizer, Think/Pair/Share

To add or subtract rational expressions with unlike denominators, find a common denominator. The easiest way to find the least common denominator is to factor the expressions. Then, the least common denominator is the product of each factor common to the expressions and any non-common factors.

EXAMPLE 4
Find the least common denominator of \( \frac{1}{x^2 - 3x - 4} \) and \( \frac{1}{x^2 - 16} \).

Step 1: Factor each denominator.
\[ x^2 - 3x - 4 = (x + 1)(x - 4) \]
\[ x^2 - 16 = (x + 4)(x - 4) \]

Step 2: Identify common factors and factors not in common.
Factors in Common: \( x - 4 \)
Factors Not in Common: \( x + 4, x + 1 \)

Step 3: Write the least common denominator.
\( (x + 4)(x + 1)(x - 4) \)

TRY THESE C
Find the least common denominator of \( \frac{1}{x^3 - 9} \) and \( \frac{1}{3x^3 - 9x} \).

\[ \frac{1}{x^3 - 9} = \frac{1}{(x - 3)(x^2 + 3x + 9)} \]
\[ \frac{1}{3x^3 - 9x} = \frac{1}{3x(x - 3)(x + 3)} \]

Now you are ready to add and subtract rational expressions with different denominators.

EXAMPLE 5
Simplify the expression. Assume no denominator is zero.

Original expression:
\[ \frac{2}{x - 2} - \frac{3}{x^2 - 2x} \]

Step 1: Factor the denominators.
\[ \frac{2}{x - 2} - \frac{3}{x(x - 2)} \]

Step 2: Find the least common denominator.
\[ x(x - 2) \]

Step 3: Multiply numerator and denominator of each term by the missing factor(s) of the least common denominator.
\[ \frac{2x}{x(x - 2)} - \frac{3}{x(x - 2)} \]

Step 4: Subtract the like fractions to find the solution.
\[ \frac{2x - 3}{x(x - 2)} \]

Math Tip
When the denominators are the same, all you have to do is add or subtract the numerators as indicated by the operation.

CONNECT TO AP
You will continue to use the skill of simplifying rational expressions in AP Calculus.

MINI-LESSON: Adding and Subtracting with Like Denominators

Use these problems if students need additional practice with adding and subtracting rational expressions with like denominators.

1. \( \frac{3 + 2x}{x + 1} + \frac{3x - 1}{x + 1} \)
2. \( \frac{3}{x - 4} - \frac{3 + 4x}{x - 4} \)
3. \( \frac{x^2}{x + 1} - \frac{1}{x + 1} \)

ACTIVITY 5.6 Continued

First Paragraph Summarize/Paraphrase/Retell

EXAMPLE 4 Note Taking, Graphic Organizer Students having difficulty may find that using a graphic organizer helps to identify the simplest common denominator. When the denominators have no common factors, the common denominator will be the product of all the denominators.

TRY THESE C Think/Pair/Share, Debriefing

EXAMPLE 5 Note Taking Ask students why the second rational expression is unchanged after the common denominator is found. Point out that this is only the case when the denominator of a rational expression is equal to the simplest common denominator.
ACTIVITY 5.6 Continued

TRY THESE D Think/Pair/Share, Simplify the Problem, Group Presentation Have students present their work to the entire class. You can circulate from group to group as they work, asking and answering questions that will assist students having difficulty.

Again, the restrictions on the variable are omitted in these problems because of the difficulty in finding them. Remind students that the restrictions are taken from the original problem, not the simplified expression. In Example 6, the restrictions are $x \neq -1, 0, 1, 2$. Notice that the simplified expression does not have the restriction $x \neq 1$. However, the original expression is not equal to the simplified expression when $x = 1$, because the original expression is undefined when $x = 1$.

First Paragraph Summarize/Paraphrase/Retell, Vocabulary Organizer, Interactive Word Wall

EXAMPLE 6 Note Taking The strategy presented to simplify complex fractions involves simplifying the numerator and denominator separately until they form a single rational expression.

Another strategy is to multiply by an appropriate form of the number 1. Select all factors to be eliminated from the denominators of the numerator and denominator of the complex fraction. In this case, you would multiply by $\frac{x + 1}{x - 1}$ as shown in the Mini-Lesson.

TRY THESE E Create Representations, Simplify the Problem, Group Presentation

A rational expression that contains rational expressions in its numerator and/or its denominator is called a complex fraction.

EXAMPLE 6 Simplify each expression. Assume no denominator is zero. Write your answers on notebook paper. Show your work.

a. $\frac{3}{x + 1} - \frac{x}{x - 1}$

b. $\frac{x^2 + 2x - 3}{x - 1}$

c. $\frac{2x^2 + 4x + 2}{x + 2}$

You can simplify complex fractions if you treat them like a division problem. Simplify the numerator and denominator as much as possible, and then write the problem using multiplication.

TRY THESE E Simplify. Assume no denominator is zero. Write your answers on notebook paper. Show your work.

a. $\frac{x^2 - 4}{2x + 2} - \frac{2}{x + 4}$

b. $\frac{1}{x + 1} - \frac{1}{x + 2}$

c. $\frac{x^2 - 2x - 1}{x(x - 2)(x + 3)}$

Suggested Assignment

CHECK YOUR UNDERSTANDING

p. 316, #6–9

UNIT 5 PRACTICE

p. 327, #43–44

MINI-LESSON: Alternate Method for Simplifying Complex Fractions

In Example 6, students can first multiply by 1 using factors that will cancel all denominators in the original complex fraction.

$$\frac{1 + \frac{1}{x + 1}}{x - \frac{1}{x - 1}} = \frac{1 + \frac{1}{x + 1}}{x - \frac{1}{x - 1}} = \frac{(x + 1)(x - 1) + (x - 1)}{x(x + 1)(x - 1) - x(x + 1)} = \frac{(x + 1)(x + 2)}{x(x + 1)(x - 2)}$$
In the graph of a rational function, a break in the graph often signals that a discontinuity has occurred. Algebraically, a discontinuity happens for values of $x$ that cause the function to be undefined and are therefore not in the domain of the function.

**EXAMPLE 7**
Identify any vertical asymptotes in the graph.

**Step 1:** Factor the numerator and denominator.

$a. f(x) = x^2 - 4$  
$x^2 + 3x - 4$

$b. f(x) = 3 - x$  
$9 - x^2$

**Step 2:** Divide out the common factors.

$a. f(x) = \frac{x^2 - 4}{x^2 + 3x - 4}$  
$\frac{(x + 2)(x - 2)}{(x + 2)(x + 3)}$

$b. f(x) = \frac{3 - x}{9 - x^2}$  
$\frac{x - 3}{x + 3}$

**Step 3:** Find the values that make the simplified denominator $= 0$.

$x + 3 = 0$ when $x = -3$  
Vertical asymptote $x = -3$

**TRY THESE F**
Identify any vertical asymptotes in the graph.

a. $f(x) = \frac{x^2 - x}{x^2 + 3x - 4}$  
hole at $x = 1$,  
Vertical Asymptote at $x = -4$

b. $f(x) = \frac{3 - x}{9 - x^2}$  
hole at $x = 3$,  
Vertical Asymptote at $x = -3$

A horizontal asymptote depends on the degrees of the numerator and denominator and describes the end behavior of a rational function.

- When the degrees are the same, the horizontal asymptote is the ratio of the leading coefficients.
- When the denominator degree is larger, the horizontal asymptote is equal to 0.
- When the numerator degree is larger, there is no horizontal asymptote.

**EXAMPLE 8**
Identify the horizontal asymptote, if any.

**a.** $f(x) = \frac{2 + x}{x^2 - 1}$  
numerator degree = 1  
denominator degree = 2  
$2 > 1$  
horizontal asymptote: $y = 0$

**b.** $f(x) = \frac{2x + 2}{x - 1}$  
numerator degree = 1  
denominator degree = 1  
lead coefficients: 2, 1  
ratio of lead coefficients: 2  
horizontal asymptote: $y = 2$

**MATH TIP**
To find the vertical asymptote of a graph, determine the values of the variables that make the function undefined when it is in simplest form.

**TECHNOLOGY TIP**
Use a graphing calculator to graph the function and visually see the breaks where the asymptotes are located.
TRY THESE G Think/Pair/Share, Look for a Pattern, Group Presentation

By the end of this lesson, students should be adept at identifying the key features of a rational function and then using those features to construct a graph.

At the Algebra 2 level, encourage students to use a graphing calculator to fill in the missing portions of the graph and confirm their work, after they have sketched in all the key features. After sufficient practice graphing rational functions with a graphing utility, students will recognize patterns and features that will allow them to sketch rational functions without needing to use a graphing utility.

Paragraph and Bulleted List Shared Reading, Questioning the Text Post the steps for graphing rational functions in a prominent location and refer to them often.

EXAMPLE 9 Graphic Organizer, Note Taking The graphic organizer shown in the example can be used to help students keep track of their work.

TRY THESE H

a. 

b. 

TRY THESE G

Identify the horizontal asymptote, if any.

\[ a. \quad f(x) = \frac{2 - x}{x + 4} \quad y = -1 \]
\[ b. \quad f(x) = \frac{x^2 - 1}{x + 3} \quad \text{none} \]
\[ c. \quad f(x) = \frac{x}{x^2 - 4} \quad y = 0 \]

Now you are ready to use your knowledge of simplifying rational expressions to help you understand and graph rational functions.

To graph rational functions, follow these steps.

- Simplify the rational function.
- Express the numerator and denominator in factored form.
- Identify vertical asymptotes.
- Identify x- and y-intercepts.
- Identify horizontal asymptote (end behavior).
- Make a sketch, using a graphing calculator as needed.

EXAMPLE 9

Analyze and graph the rational function \( f(x) = \frac{x^3 + 5x - 14}{x^2 - 4} \).

Simplify.

\[
\frac{x^3 + 5x - 14}{x^2 - 4} = \frac{(x + 7)(x - 2)}{(x + 2)(x - 2)} = \frac{x + 7}{x + 2}
\]

Identify vertical asymptote.

\( x + 2 = 0, \text{ so } x = -2 \)

Identify intercepts.

\( x\)-intercept: \( x + 7 = 0, \text{ so } x = -7 \)
\( y\)-intercept: \( f(0) = \frac{0 + 7}{0 + 2} = 3.5 \)

Identify horizontal asymptote.

numerator degree = 1
denominator degree = 1
lead coefficients: 1, 1
ratio of lead coefficients: 1
horizontal asymptote: \( y = 1 \)

Graph.
EXAMPLE 10
Analyze and graph the rational function \( f(x) = \frac{2}{x - 2} - \frac{3}{x^2 - 2x} \).

**Simplify.**

\[
\frac{2}{x - 2} - \frac{3}{x^2 - 2x} = \frac{2(x - 2)}{x(x - 2)} - \frac{3x}{x(x - 2)} = \frac{2x - 4 - 3x}{x(x - 2)} = \frac{-x - 4}{x(x - 2)}
\]

**Identify vertical asymptotes.**

\( x = 0 \) and \( x = 2 \) so vertical asymptotes are \( x = 0 \) and \( x = 2 \)

**Identify intercepts.**

- \( x \)-intercept: 2
- \( y \)-intercept: none, because \( f(x) \) is undefined when \( x = 0 \)

**Identify horizontal asymptote.**

- Numerator degree = 1
- Denominator degree = 2
- Horizontal asymptote is \( y = 0 \)

**Graph.**

![Graph of the rational function](image)

TRY THESE H
Analyze and graph each rational function. Write your answers on grid paper. Show your work. See side column, page 314 for graphs.

a. \( f(x) = \frac{x^2 - 4}{x^3 - 3x^2 - 10x} \)
   - \( x \)-intercept: (2, 0), no \( y \)-intercept, hole at \( x = -2 \), vertical asymptote at \( x = 0 \), \( x = 5 \), horizontal asymptote at \( y = 0 \)

b. \( f(x) = \frac{1}{x + 1} - \frac{2}{x + 3} \)
   - \( x \)-intercept: \( 1, 0 \), \( y \)-intercept: \( 0, \frac{1}{3} \), no holes, vertical asymptote at \( x = -3 \), \( x = -1 \), horizontal asymptote at \( y = 0 \).

**Math Tip**

To determine when a sum or difference of rational expressions is 0 or undefined, it helps to combine them into a single expression first.

ACTIVITY 5.6 Continued

EXAMPLE 10 Graphic Organizer, Note-Taking Again, a graphic organizer is used to help students with their work. In this example, students must perform an operation on rational expressions before graphing the function.

**TRY THESE H** Graphic Organizer, Indentify a Subtask, Create Representations, Group Presentation, Debriefing

**Differentiating Instruction**

Students having difficulty may need to work with simpler examples like the ones below. Since the focus of this activity is graphing rational functions, it may be helpful to substitute these problems if simplifying and then graphing rational functions poses too many difficulties for students.

**Alternate Examples**

Analyze and graph each rational function.

1. \( f(x) = \frac{x}{x^2 - 4} \)
   - \( x \)-intercept: 0
   - Vertical Asymptote: \( x = 2 \) and \( x = -2 \)
   - Graph shown below.

2. \( f(x) = \frac{x + 3}{x^2 + 4x + 3} \)
   - No \( x \)-intercept, \( y \)-intercept: 1
   - Vertical Asymptote: \( x = -1 \)
   - Horizontal Asymptote: \( y = 0 \)
   - Graph shown below.

**Suggested Assignment**

CHECK YOUR UNDERSTANDING
p. 316, #10–13

UNIT 5 PRACTICE
p. 327, #45
CHECK YOUR UNDERSTANDING

1. \( \frac{2x + 1}{x - 2} \), \( x \neq -2, 2 \)
2. \( \frac{4 - x}{x(x + 2)} \), \( x \neq 0, -4, -2 \)
3. \( \frac{x + 6}{x - 20} \)
4. \( \frac{1}{x - 1} \)
5. \( \frac{(x + 2)(x - 3)}{2(x - 2)} \)
6. \( 5(x + 2)^2 \)
7. \( (x - 3)^2 (x + 10) \)
8. \( -x^2 - x - 2 \)
9. \( \frac{x + 2}{(x - 4)(x + 1)(x - 1)} \)
10a. hole: \( x = -5 \)
    Vertical Asymptote: \( x = 7 \)
    Horizontal Asymptote: \( y = 1 \)
10b. hole: \( x = -2 \)
    Vertical Asymptote: \( x = 2 \)
    Horizontal Asymptote: \( y = 0 \)
11. \( x \)- and \( y \)-intercept 0
    hole: \( x = -2 \)
    Vertical Asymptote: \( x = 3 \)
    Horizontal Asymptote: \( y = 1 \)
12. \( y \)-intercept \( -1 \), Vertical
    Asymptote: \( x = -1 \), \( x = 1 \)
    Horizontal Asymptote: \( y = 1 \)
13. Answers may vary.

CHECK YOUR UNDERSTANDING

Write your answers on notebook paper or grid paper. Show your work.

Simplify. Identify any restrictions on \( x \).
1. \( \frac{2x^2 + 5x + 2}{x^2 - 4} \)
2. \( \frac{16 - x^3}{x^3 + 6x^2 + 8x} \)

Perform the indicated operation. Assume no denominator is equal to zero.
3. \( \frac{x^2 - 5x - 6}{x^2 - 12x + 36} \cdot \frac{x^2 - 36}{x^2 - 19x - 20} \)
4. \( \frac{2x^2 + 3x + 1}{x^2 - 1} \cdot \frac{2x + 1}{4x^2 + 4x + 1} \)
5. \( \frac{3x^2 + 4x - 4}{2x - 4} \div \frac{9x^2 - 4}{3x^2 - 7x - 6} \)

Find the simplest common denominator.
6. \( \frac{1}{5x + 10} \) and \( \frac{2}{x^2 + 4x + 4} \)
7. \( \frac{1}{x - 3} \) and \( \frac{2x}{x^3 - 6x + 9} \) and \( \frac{2x}{x^3 + 7x - 30} \)

Perform the indicated operation. Assume no denominator is equal to zero.
8. \( \frac{2}{x + 3} - \frac{x}{x - 1} \)
9. \( \frac{2}{x^2 - 3x - 4} \div \frac{1}{x^2 - 1} \)
10. For each function, identify any vertical asymptotes and horizontal asymptotes.
    a. \( f(x) = \frac{x^2 - 25}{x^2 - 2x - 35} \)
    b. \( f(x) = \frac{2x + 4}{x^2 - 4} \)

Analyze and graph each rational function.
11. \( f(x) = \frac{x^2 + 2x}{x^2 - x - 6} \)
12. \( f(x) = \frac{x}{x + 1} + \frac{1}{x - 1} \)
13. **Mathematical Reflection**

What have you learned about simplifying rational expressions and graphing rational functions as a result of this activity?
Jesse pitches for the baseball team and wants to improve his batting average before the county all-stars are selected. To date, he has 10 hits out of 40 times at the bat.

1. Batting average is the ratio of hits to at-bats. Write a ratio that represents Jesse’s current batting average for this season and express the ratio in decimal form.
   \[
   \frac{10}{40} = 0.250
   \]

Jesse wants to improve his batting average to at least 0.320. If he gets a hit every time he bats, then his new batting average would be \( \frac{10 + x}{40 + x} \) where \( x \) is the number of future hits in as many times at-bat.

2. Write an equation to determine how many consecutive hits he needs to bat 0.320.
   \[
   \frac{10}{40} + x = 0.320
   \]

To solve equations like the one you wrote in Item 2, multiply by an expression that eliminates all the denominators.

**EXAMPLE 1**

Solve \( \frac{x^2 - 4}{x + 1} = x + 5 \).

Original equation, undefined at \( x = -1 \)

\[
\frac{x^2 - 4}{x + 1} = x + 5
\]

Step 1: Multiply both sides by \((x + 1)\)

\[
(x + 1) \left( \frac{x^2 - 4}{x + 1} \right) = (x + 5)(x + 1)
\]

Step 2: Solve for \( x \).

\[
x^2 - 4 = x^2 + 6x + 5
\]

\[
4 = 6x + 5
\]

\[
x = -\frac{1}{3}
\]

Step 3: Check to see if the original equation is undefined at the solution.

3. Solve the equation you wrote in Item 2 to find the number of consecutive hits that Jesse needs to increase his batting average.

\[
x = 4.118
\]

Jesse would need 5 consecutive hits.

**CONNECT TO MEASUREMENT: Ratios vs. Rates**

There has been much discussion in mathematics education over the difference between ratios and rates. Some definitions and textbooks separate them as distinct terms, while others state that a rate is simply a special type of ratio.

- ratio: any comparison of two numbers or measurements
- rate: a special ratio in which the two terms have different units

Most mathematicians agree that ratio is a more general term that encompasses rates, and is simply a relationship between two quantities. In this activity, ratio is used as a generic term, while rate can be applied to ratios that involve two different kinds of units.
ACTIVITY 5.7 Continued

EXAMPLE 2  Note Taking, This problem is more complicated than Example 1 because it involves subtraction and multiple rational expressions.

TRY THESE A  Think/Pair/Share, Group Presentation  Have students work in groups and give them the opportunity to check their solutions.

Paragraph Vocabulary Organizer, Interactive Word Wall  Extraneous solutions may occur when you multiply or divide both sides of the original equation by the simplest common denominator. The rational equation is transformed into a polynomial equation. Solutions to the resulting polynomial equation are not always solutions to the original rational equation because multiplying by the fraction \( \frac{f(x)}{g(x)} \) as you rewrite the equation leads to extraneous solutions when \( f(x) = 0 \).

4 Identify a Subtask

5 Guess and Check  Be sure that students check their solutions in the original equation to find any extraneous solutions.

TRY THESE B  Think/Pair/Share, Debriefing

SUGGESTED LEARNING STRATEGIES: Note Taking, Think/Pair/Share, Group Presentation, Vocabulary Organizer, Interactive Word Wall, Identify a Subtask, Guess and Check

EXAMPLE 2
Solve \( \frac{2}{x} - \frac{1}{x+2} = \frac{3}{x} \).

Original equation, undefined at \( x = 0 \) and \( x = -2 \)

Step 1: Multiply both sides by \( x(x + 2) \) to cancel the denominators.

\[ \frac{2}{x}(x(x + 2)) - \frac{1}{x+2}(x(x + 2)) = \frac{3}{x}(x + 2) \]

\[ x(x + 2)(\frac{2}{x} - \frac{1}{x+2}) = (\frac{3}{x})(x + 2) \]

Step 2: Solve for \( x \).

\[ 2(x + 2) - 1(x) = 3(x + 2) \]

\[ 2x + 4 - x = 3x + 6 \]

\[ 2x + 4 - 3x - 6 = 0 \]

\[ -2x = 2 \]

\[ x = -1 \]

TRY THESE A

Solve each equation and check your solution.

a. \( \frac{x + 4}{x + 5} = \frac{3}{5} \)

\[ x = -2.5 \]

b. \( \frac{2x}{x + 2} - \frac{x}{x - 1} = 1 \)

\[ x = \frac{2}{3} \]

When checking your solutions, substitute the solution into the original equation.

When solving a rational equation, it is possible to introduce an extraneous solution. An **extraneous solution** is not valid in the original equation although it satisfies the polynomial equation that results when you multiply by the simplest common denominator.

4. Solve the equation \( \frac{1}{x} - \frac{2x}{x + 2} = \frac{x - 6}{x(x + 2)} \)

\[ x = 2 \]

5. Identify any extraneous solutions to the equation in Item 4. \( x = -2 \) is an extraneous solution.

TRY THESE B

Solve each equation. Identify any extraneous solutions. Write your answers on notebook paper. Show your work.

a. \( \frac{x}{x - 1} = \frac{1}{x - 1} + \frac{2}{x} \)

\[ x = 2; \ x = 1 \text{ is an extraneous solution.} \]

b. \( \frac{1}{2} - \frac{x - 1}{x^2 + x} = \frac{x - 1}{x + 1} \)

\[ x = 2; \ x = -1 \text{ is an extraneous solution.} \]
MINI-LESSON: Alternative Method for Setting Up a Work Problem

This method focuses on the amount of work each person contributes to the completed job. If Jesse does \( \frac{1}{5} \) of a job in an hour, then during the 2-hour job, he will do \( 2 \left( \frac{1}{5} \right) \) or \( \frac{2}{5} \) of the work. In 2 hours, Cody will do \( \frac{2}{7} \) of the job.

Together they will complete 1 job.

\[
\begin{align*}
\text{Jesse’s share} & \quad + \quad \text{Cody’s share} \quad = \quad \text{the completed job} \\
\frac{1}{5} \left( \frac{1}{2} \right) & \quad + \quad \frac{2}{7} \left( \frac{1}{2} \right) \quad = \quad 1
\end{align*}
\]

This equation is equivalent to the one in Item 9.
TEACHER TO TEACHER

The focus of the second half of this activity is solving rational inequalities. A graphical/numerical approach is introduced that relies on graphing calculator technology.

ACTIVITY 5.7 Continued

12. Questioning the Text, Identify a Subtask

a. Look for a Pattern Make the connection between the graph and the table explicit to students. A more analytical approach is used that relies on analyzing the sign of the rational expression between its zeros and undefined values.

b. Quickwrite, Debriefing

Solutions to these problems are expressed using inequalities. This might be something you need to review with your students. An inequality like \( x < 3 \) could also be written as \( -\infty < x < 3 \). Always use a strict inequality when the endpoint is positive or negative infinity.

TECHNOLOGY Tip

- In order to see all of the sign changes, it may be necessary to increment the table by a number smaller than 1.
- Make sure rational expressions are entered into the calculator correctly—the entire denominator and the entire numerator of the rational function must be enclosed in parentheses for the calculator to evaluate the function using the correct order of operations.
- Most graphing calculators will indicate points where a function is undefined by leaving the y-value blank or displaying an error message.
- Try to determine ahead of time a reasonable window that will allow students to see zeros and asymptotes.

The rational inequality shown below can be solved graphically or numerically.

\[
\frac{x^2 - 1}{x^2 - x - 12} < 0
\]

12. First, factor the left side of the inequality and determine the zeros and the values of \( x \) that are not in the domain of the function.

\[
(x + 1)(x - 1) < 0; \text{ vertical asymptotes at } x = 4, -3; \text{ zeroes at } x = \pm 1
\]

13. The graph of the left side of the inequality is shown below. The table shows the \( x \) - and \( y \)-coordinates and the sign of \( y \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
<th>sign</th>
</tr>
</thead>
<tbody>
<tr>
<td>-5</td>
<td>1.333</td>
<td>+</td>
</tr>
<tr>
<td>-4</td>
<td>1.875</td>
<td>+</td>
</tr>
<tr>
<td>-3</td>
<td>undefined</td>
<td></td>
</tr>
<tr>
<td>-2</td>
<td>-0.5</td>
<td>-</td>
</tr>
<tr>
<td>-1</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0.083</td>
<td>+</td>
</tr>
<tr>
<td>0.5</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>-0.3</td>
<td>-</td>
</tr>
<tr>
<td>3</td>
<td>-1.333</td>
<td>-</td>
</tr>
<tr>
<td>4</td>
<td>undefined</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>+</td>
</tr>
</tbody>
</table>

a. Identify the intervals of \( x \) where the graph is below the \( x \)-axis.

\(-3 < x < -1, 1 < x < 4\)

b. Look back to the original inequality. Why would the intervals of \( x \) where the graph is below the \( x \)-axis be the solutions to the inequality?

The rational expression on the left hand side of the equation is less than zero when the \( y \)-values are negative. This happens in quadrants III and IV or when the graph is below the \( x \)-axis.
### EXAMPLE 3

Solve the inequality \(-\frac{x^2 - 1}{x^2 - 2x - 8} \leq 0\).

**Factor:** \((x + 1)(x - 1) \leq 0\)

**Zeros of the numerator at** \(x = 1\) and \(-1\)

**Zeros of the denominator (where function is undefined) at** \(x = -2\) and 4

The zeros, in order from least to greatest are: \(-2, -1, 1, 4\)

Pick and test one value in each interval: \(-3, -1.5, 0, 2,\) and 5

<table>
<thead>
<tr>
<th>Interval</th>
<th>(-3)</th>
<th>(-1.5)</th>
<th>0</th>
<th>2</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sign</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>+</td>
</tr>
</tbody>
</table>

Continue this process and record the results in a table.

The solution is the intervals of \(x\) where the inequality was less than or equal to 0 (recall the “\(\leq\) 0” in the original inequality). So, \(x\)-values of the numerator zeros are included in the solution.

**Solution intervals:** \(-2 \leq x \leq -1\) or \(1 \leq x < 4\)

Graph the solution on a number line.

---

**Math Tip**

To use this method, the inequality must be set up so that one side is a single rational expression and the other side is 0. If it is not, first transform the inequality into that form.

---

**Math Tip**

You do not need to evaluate the inequality completely. Simply plug in the test values and figure out the sign of each factor and then the overall sign of the rational expression.

---

There is a reason why the use of test values works to identify the solution sets, after you have displayed all zeros and all values not in the domain of the function. This method of test values works because, by the Intermediate Value Property of Polynomials (deducible from the Fundamental Theorem of Algebra), if a polynomial has values \(P(a)\) and \(P(b)\) that have different signs, there must be a value \(c\) between \(a\) and \(b\) at which \(P(c) = 0\). Consequently, if all the zeros of the numerator polynomial and the denominator polynomial have been located, it follows that the sign cannot change in intervals formed by these values.
### Suggested Assignment

**CHECK YOUR UNDERSTANDING**

p. 322, #7–9

**UNIT 5 PRACTICE**

p. 327, #50

### CHECK YOUR UNDERSTANDING

1. \( x = -5, 2 \)
2. no solution
3. \( x = 4; x = 1 \) is extraneous.
4. \( x = -3 \)
5. The chemist should add about 28.6 units to the solution.
6. It will take about 1.3 hours.
7. \(-3 < x < -1 \) or \( 3 < x < 5 \)
8. \( x \leq -3 \) or \(-1 < x < 1 \)
9. Answers will vary.

### TRY THESE C

Solve each inequality.

a. \( \frac{x^2 - 5x - 6}{x^2 - 4x + 3} > 0 \)
   - zero at \( x = 0 \) and \(-1 \), undefined at \( x = 1 \) and 3,
   - solution: \( x < -1 \) or \( 1 < x < 3 \) or \( x \geq 6 \)

b. \( \frac{\frac{1}{2} - \frac{2}{x + 2}}{x^2 - 1} < 0 \)
   - zero at \( x = 2 \), undefined at \( x = 0 \) and \(-2 \),
   - solution: \(-2 < x < 0 \) or \( x > 2 \)

### CHECK YOUR UNDERSTANDING

Write your answers on notebook paper or grid paper. Show your work.

Solve each equation. Identify any extraneous solutions.

1. \( \frac{x + 3}{2} = \frac{5}{x} \)
2. \( \frac{2}{x} + \frac{3}{x} = \frac{5}{x + 1} \)
3. \( \frac{x - 3}{x - 1} - \frac{2}{x + 1} = \frac{x - 5}{x^2 - 1} \)
4. \( \frac{1}{x - 3} = \frac{x}{9 - 3x} \)
5. A chemist has 100 units of a 10% solution and wants to strengthen it to 30%. How much pure chemical should be added to the original solution to achieve the desired concentration?

6. Raj, Ebony, and Jed paint houses during the summer. Raj takes 5 hours to paint a room by himself while it takes Ebony 4 hours and Jed 3 hours. How long will it take them if they work together?

Solve each inequality graphically or numerically.

7. \( \frac{x^2 - 9}{x^2 - 4x - 5} < 0 \)
8. \( \frac{1}{x + 1} = \frac{2}{x - 1} \)
9. **MATHEMATICAL REFLECTION** What have you learned about solving rational equations and inequalities as a result of this activity?
Rational Equations and Functions

**PLANNING A PROM**

1. Sketch the graph of the rational function \( f(x) = \frac{x - 1}{x + 1} \). Identify the key features, such as asymptotes and intercepts. Then describe the graph as a transformation of the parent function \( f(x) = \frac{1}{x} \).

2. Simplify the rational expression \( \frac{x - 2}{x + 2} - \frac{x + 1}{x - 1} \). Assume no denominator is zero.

3. Solve the equation \( \frac{2}{x} - \frac{2}{x + 2} = 8 \) in two ways and discuss the advantages and disadvantages of the methods you selected.

4. The prom committee is planning this year’s prom. The costs are stated in the table below.

<table>
<thead>
<tr>
<th>Item</th>
<th>Cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>Music</td>
<td>$800</td>
</tr>
<tr>
<td>Decorations</td>
<td>$500</td>
</tr>
<tr>
<td>Ballroom</td>
<td>$900</td>
</tr>
<tr>
<td>Catered Dinner</td>
<td>$25/person</td>
</tr>
</tbody>
</table>

The committee must set aside 10 free prom tickets for a drawing the principal wants to have for students enrolled in Advanced Placement® classes. The ballroom can only hold 300 students, and the prom committee has not decided whether they want to have dinner catered at the prom. Based on past experience, the committee knows students are not willing to pay more than $20 per ticket if food will not be provided and $35 per ticket if food is provided.

Write a proposal for setting the prom ticket price. Be sure to include these items:
- the proposed ticket price
- the number of tickets that must be sold to break even
- the amount of money that will be made
- mathematics to support your reasoning

---

**Answer Key**

1. \( x \)-intercept 1 and \( y \)-intercept \(-1\), Vertical Asymptote: \( x = -1 \), Horizontal Asymptote: \( y = 1 \). This graph is the parent function stretched vertically by a factor of 2, reflected about the \( x \)-axis, and translated 1 unit left and 1 unit up.

2. This Item focuses on simplifying a rational expression.

\[
\frac{x - 2}{x + 2} - \frac{x + 1}{x - 1} = \frac{(x - 2)(x - 1) - (x + 1)(x + 2)}{(x + 2)(x - 1)} = \frac{-6x}{(x + 2)(x - 1)}
\]

---

**Embedded Assessment 2**

**Activity Focus**
- Analyzing and graphing rational functions
- Solving rational equations
- Rational models and applications

**Materials**
- Graphing calculator

**Teacher to Teacher** You may wish to assign Item 4 as a take-home quiz to give students the opportunity to plan a thoughtful, well-written response to the Item.

This Item asks students to graph a rational function and then describe the graph as a transformation of the parent rational function. Students may do this algebraically using long division or from the graphs they create.

**Teacher to Teacher** You may wish to read through the rubric on the next page with students and discuss the differences in the expectation levels. Make sure students understand the meanings of any terms used.
Embedded Assessment 2

3 Students will most likely solve this equation algebraically and graphically using a graphing calculator to find the intersection points.

Solve algebraically.

\[
\frac{2}{x} - \frac{2}{x + 2} = 8
\]

Multiply by \(x(x + 2)\).

\[
2(x + 2) - 2(x) = 8(x)(x + 2)
\]

\[
4 = 8x^2 + 16x
\]

\[
8x^2 + 16x - 4 = 0
\]

\[
x = \frac{-2 \pm \sqrt{6}}{2}
\]

Solve graphically using a calculator.

4 Students will need time to read and process this Item and to prepare a response. The Item revisits a context similar to Activity 5.4 Summer Camp, but this time students are planning a prom. This Item requires them to create rational functions from verbal models, solve rational equations/inequalities, graph rational functions, make decisions based on mathematical information and communicate their understanding in written form.

4 Answers may vary but should include the following information. Answers should include multiple representations to justify chosen ticket price.

<table>
<thead>
<tr>
<th>Math Knowledge #1, 2, 3, 4</th>
<th>Exemplary</th>
<th>Proficient</th>
<th>Emerging</th>
</tr>
</thead>
<tbody>
<tr>
<td>The student:</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>• Identifies four key features of the graph. (1)</td>
<td>The student:</td>
<td>The student:</td>
<td></td>
</tr>
<tr>
<td>• Simplifies the rational expression correctly. (2)</td>
<td>• Identifies only three key features of the graph.</td>
<td>• Identifies at least one key feature of the graph.</td>
<td></td>
</tr>
<tr>
<td>• Solves the equation correctly in two ways. (3)</td>
<td>• Uses the correct method to simplify the expression, but makes a computational error.</td>
<td>• Does not simplify the expression correctly.</td>
<td></td>
</tr>
<tr>
<td>• Uses the correct mathematics to support his/her reasoning. (4)</td>
<td>• Solves the equation correctly in only one way.</td>
<td>• Does not solve the equation correctly.</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Problem Solving #4</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>The student correctly determines the number of tickets that must be sold to break even and the amount of money that will be made for the proposed ticket price. (6)</td>
<td>The student correctly determines the number of tickets that must be sold to break even or the amount of money that will be made for the proposed ticket price.</td>
<td>The student correctly determines neither the number of tickets that must be sold nor the amount of money that will be made.</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Representations #1</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>The student sketches a correct graph. (1)</td>
<td>The student sketches a partially correct graph.</td>
<td>The student sketches an incorrect graph.</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Communication #1, 3, 4</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>The student:</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>• Gives a complete description of the transformation. (1)</td>
<td>The student:</td>
<td>The student:</td>
<td></td>
</tr>
<tr>
<td>• Gives a complete discussion of the advantages and disadvantages of the methods selected. (3)</td>
<td>• Gives an incomplete description of the transformation.</td>
<td>• Gives an incomplete description of the transformation.</td>
<td></td>
</tr>
<tr>
<td>• Writes a proposal that includes the four indicated items. (4)</td>
<td>• Gives an incomplete description of the advantages and disadvantages of the methods selected.</td>
<td>• Does not discuss the advantages and disadvantages of the methods selected.</td>
<td></td>
</tr>
</tbody>
</table>

Ticket price function without food

\[
f(x) = \frac{2200}{x - 10}
\]

Ticket price function with food

\[
f(x) = \frac{2200 + 25x}{x - 10}
\]

The prom committee must sell at least 120 tickets at $20 to break even if they don’t provide food. The prom committee must sell at least 255 tickets at $35 to break even if they provide food.

If a lower ticket price is selected, then the committee will need to sell more tickets to break even. Evaluate \(f(300)\) to determine the minimum per person ticket price. If 300 people attended the prom, then the ticket price without food could be $7.59 per person and with food it could be $33.45 per person.
UNIT 5 PRACTICE

Activity 5.1

1. B
2a. \( f^{-1}(x) = \frac{5}{9}(x - 32) \)
   b. Answers will vary. One ordered pair solution is shown: \((0, 32)\) is a solution for \( f^{-1}(x) = \frac{5}{9}(x - 32) \) because \(0 = \frac{5}{9}(32 - 32)\).
   c. \((0, 32)\) means 0 degrees Celsius is 273 degrees Fahrenheit. \((32, 0)\) mean 32 degrees Fahrenheit is 0 degrees Celsius. In general, on function \(f\) the points are \((\text{Celsius, Fahrenheit})\) and on \(f^{-1}\) the points are \((\text{Fahrenheit, Celsius})\).
   d. The intersection point of the lines \(y = \frac{5}{9}(x - 32)\) and \(y = \frac{9}{5}x + 32\) is \((-40, -40)\). Substituting \(-40\) into each temperature conversion formula gives an output of \(-40\).

3. \( y = \sqrt{x - 3} \rightarrow x = \sqrt{y - 3} \)
   \(f^{-1}: D = x \geq 0\)
   \(R = y \geq 3\)
   \(f: D = x \geq 3\)
   \(R = y \geq 0\)

4. B
5a. \( f(x) = \frac{9}{5}(x - 273.15) + 32 \) where \(x\) is in Kelvin.
   b. \( k(x) = \frac{9}{5}(x - 32) + 273.15 \) where \(x\) is degrees Fahrenheit. Function \(k\) is the inverse of \(f\).
   c. \(-459.67^\circ\ F\)
   d. 373.15 K

Activity 5.2

6. Reflection across x-axis, vertical stretch by a factor of 4, vertical translation of 2 up. Domain: \(x \geq 0\). Range: \(y \leq 2\).

7. Horizontal translation 2 units to the left and vertical translation 3 units up. Domain: \(x \geq -2\). Range \(y \geq 3\).

Activity 5.2, 10–12 and Activity 5.3, 13–17: see page 326.
Activity 5.3

13. \( \frac{1}{3} \)
14. \( 2x^3/2 \)
15. 36
16. \( x = 625 \)
17. \( x = -3 \)
18. \( x = 3 \)
19. \( 5\sqrt{125} \)
20. \( x = 0 \) and 1. The solutions are the intersection points of the graphs of \( y = x^3 \) and \( y = x^2 \).
21. \( x > 5.196 \) or \( x < -5.196 \)

Activity 5.4

22. \( f(x) = 10,000 + 12x \)
23. \( c(x) = \frac{10,000 + 12x}{x} \)
24. [Graph of Cost Per Condo vs. Number of Kitty Condos]

25. $32

26. Solve the equation \[ \frac{10,000 + 12x}{x} = 13 \] \( x = 10,000 \)

27. C
28. B
29. C
30. [Graph of function]

Activity 5.5

31. Given the inverse variation \( y = \frac{10}{x} \), what is the constant of variation?
   a. \( k = -10 \)
   b. \( k = 1 \)
   c. \( k = 10 \)
   d. \( k = 100 \)

32. If \( y \) varies inversely as \( x \), and \( y = 8 \) when \( x = 40 \), which equation models this situation?
   a. \( y = \frac{5}{x} \)
   b. \( y = \frac{32}{x} \)
   c. \( y = \frac{48}{x} \)
   d. \( y = \frac{320}{x} \)

33. Evan’s video game scores vary inversely as the time spent playing. If he scores 1000 points after playing for 1 hour, how much will he score after playing 3.5 hours?

34. The number of pages that Emma reads varies directly as the time spent reading. If Emma reads 120 pages in 1.5 hours, how many pages does she read in 45 minutes?

35. Write a function that is \( f(x) = \frac{1}{x} \) translated up 2 units and 6 units to the left.

36. Which function is a vertical translation and a vertical stretch of \( f(x) = \frac{1}{x} \)?
   a. \( f(x) = \frac{x}{2} \)
   b. \( f(x) = \frac{1}{x+2} \)
   c. \( f(x) = \frac{2}{x} + 1 \)
   d. \( f(x) = \frac{1}{2} + x \)

Solve the equation \( \sqrt{x} - \sqrt{3} = \sqrt{x} + \sqrt{3} \)

20. Use what you know about the graphs of power functions to solve this equation: \( x^3 = x^2 \)
21. Solve graphically or analytically: \( 2x > 6 \)

Activity 5.4

KitKat Kondos makes kitty condos. They have $10,000 in fixed operating costs and each kitty condo costs $12 to make.

22. Write a function that represents the cost of making \( x \) kitty condos.
23. Write a rational function that represents the cost per condo of \( x \) kitty condos.
24. Graph the cost per condo function.
25. What is the cost per condo for 500 kitty condos?
26. If the cost per condo was $13, how many condos did the company make?
27. Which function is an example of a rational function?
   a. \( f(x) = \frac{3 - 3x^2}{x - 2} \)
   b. \( f(x) = \frac{x - 3x^2}{2} \)
   c. \( f(x) = \frac{-3x^2}{x - 2} \)
   d. \( f(x) = x - 3x^2 \)

Use \( f(x) = \frac{x + 2}{2x + 1} \) to answer Items 28–29.

28. What is the vertical asymptote of \( f \)?
   a. \( x = -2 \)
   b. \( x = -0.5 \)
   c. \( x = 0.5 \)
   d. \( x = 2 \)
29. What is the horizontal asymptote of \( f \)?
   a. \( y = -2 \)
   b. \( y = -0.5 \)
   c. \( y = 0.5 \)
   d. \( y = 2 \)
30. Graph the rational function \( f(x) = \frac{x - 3}{x + 3} \)

Activity 5.5

31. C
32. D
33. Evan will score 286 points after 3.5 hours of playing.
34. Emma will read 60 pages in 45 minutes.
35. \( f(x) = \frac{1}{x + 6} + 2 \)
36. C
Practice

UNIT 5

ACTIVITY 5.6

Use \( f(x) = \frac{2}{x+1} - 5 \) to answer Items 37–38.

37. What is the vertical asymptote of \( f \)?
   a. \( x = -5 \)
   b. \( x = -1 \)
   c. \( x = 1 \)
   d. \( x = 2 \)

38. What is vertical stretch of \( f \)?
   a. 2
   b. 1
   c. \(-5\)
   d. none

39. Graph the rational function \( f(x) = \frac{-2}{x+5} \).

ACTIVITY 5.6

40. What are the restrictions on \( x \) in the rational expression \( \frac{16-x^2}{4x+16} \)?
   a. none
   b. \( x \neq 0 \)
   c. \( x \neq -4 \)
   d. \( x \neq \pm 4 \)

41. Simplify \( \frac{16-x^2}{4x+16} \). Assume no denominator equals zero.
   a. \( 4-x \)
   b. \( 1-\frac{x}{4} \)
   c. \( \frac{4-x}{4} \)
   d. \( \frac{4-x}{x+4} \)

42. Divide \( \frac{4x^2+4}{x^2-1} \). Assume no denominator is zero.
   a. 4
   b. \( \frac{4}{x} \)
   c. \( 4-x \)
   d. \( \frac{4x+4}{x} \)

43. Simplify \( \frac{6}{x-6} + \frac{x}{x+6} \).
   a. \(-1\)
   b. \( \frac{x+6}{x-6} \)
   c. \( \frac{1}{x-6} \)
   d. \( \frac{x^2+36}{(x-6)(x+6)} \)

44. Simplify \( \frac{1}{x+1} - \frac{1}{x+3} \). Assume no denominator equals 0.
   a. 1 \( \frac{1}{x} \)
   b. \( \frac{1}{x} \)
   c. \( 1-x \)
   d. \( \frac{x+1}{1-x} \)

45. Analyze and graph \( f(x) = \frac{x^2-36}{x^2-9x-6} \).

ACTIVITY 5.7

46. Solve \( \frac{x+3}{x-6} = \frac{2}{5} \).
   a. \( x = -1 \)
   b. \( x = -9 \)
   c. \( x = 9 \)
   d. no solution

47. Solve \( \frac{-2x}{x-1} + \frac{x-3}{x-1} = 2 \).
   a. \( x = 1 \)
   b. \( x = 0.5 \)
   c. \( x = 1, 0.5 \)
   d. no solution

48. The resistance, \( R_{eq} \), of a circuit with 2 resistors in parallel is given in Ohms by \( \frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} \).
   If the resistors come in 5 ohm increments, find a pair of resistors so the circuit has a resistance of 6 ohms.

49. When Joe and Jon work together, they can wire a room in 8 hours. Working alone, Joe needs 12 hours to wire a room. How long would it take Jon working alone to wire a room?

50. Solve each inequality numerically or graphically.
   a. \( \frac{x+3}{x-1} \leq 0 \)
   b. \( x^2-x-6 > 0 \)

Activity 5.6

40. C
41. C
42. B
43. D
44. D
45. hole: (6, \( \frac{12}{5} \)),
   x-intercept -6,
   y-intercept 6,
   Vertical Asymptote: \( x = -1 \),
   Horizontal Asymptote: \( y = 1 \)

Activity 5.7

46. b
47. d
48. A 10 ohm and a 15 ohm resistor should be used.
49. It will take Jon 24 hours.
50a. \( x \leq -3 \) or \(-1 < x < 1 \)
50b. \( x < -3 \) or \(-2 < x < 1 \)
   or \( x > 3 \)
Reflection

**Student Reflection**
Discuss the essential questions with students. Have them share how their understanding of the questions has changed through studying the concepts in the unit.

Review the academic vocabulary. You may want students to revisit the graphic organizers they have completed for academic vocabulary terms and add other notes about their understanding of terms.

Encourage students to evaluate their own learning and to recognize the strategies that work best for them. Help them identify key concepts in the unit and to set goals for addressing their weaknesses and acquiring effective learning strategies.

**Teacher Reflection**
1. Of the key concepts in the unit, did any present special challenges for students?
2. How will you adjust your future instruction for students/activities?
3. Which strategies were most effective for facilitating student learning?
4. When you teach this unit again, what will you do differently?

An important aspect of growing as a learner is to take the time to reflect on your learning. It is important to think about where you started, what you have accomplished, what helped you learn, and how you will apply your new knowledge in the future. Use notebook paper to record your thinking on the following topics and to identify evidence of your learning.

**Essential Questions**
1. Review the mathematical concepts and your work in this unit before you write thoughtful responses to the questions below. Support your responses with specific examples from concepts and activities in the unit.
   - Why is it important to consider the domain and range of a function?
   - How are inverse functions useful in everyday life?

**Academic Vocabulary**
2. Look at the following academic vocabulary words:
   - complex fraction
   - horizontal asymptote
   - inverse variation
   - one-to-one function
   - power function
   - rational exponent
   - rational function
   - vertical asymptote

Choose three words and explain your understanding of each word and why each is important in your study of math.

**Self-Evaluation**
3. Look through the activities and Embedded Assessments in this unit. Use a table similar to the one below to list three major concepts in this unit and to rate your understanding of each.

<table>
<thead>
<tr>
<th>Unit Concepts</th>
<th>Is Your Understanding Strong (S) or Weak (W)?</th>
</tr>
</thead>
<tbody>
<tr>
<td>Concept 1</td>
<td></td>
</tr>
<tr>
<td>Concept 2</td>
<td></td>
</tr>
<tr>
<td>Concept 3</td>
<td></td>
</tr>
</tbody>
</table>

a. What will you do to address each weakness?
b. What strategies or class activities were particularly helpful in learning the concepts you identified as strengths? Give examples to explain.

4. How do the concepts you learned in this unit relate to other math concepts and to the use of mathematics in the real world?
1. Consider the functions: \( c(x) = \frac{1}{\sqrt{x - 4}} \) and \( d(x) = 2x \). What is the domain of \( c(d(x)) \) over the set of real numbers?
   A. \( \{x : x \in \text{real numbers}\} \)
   B. \( \{x : x > 4\} \)
   C. \( \{x : x > 2\} \)
   D. \( \{x : x > 0\} \)

2. What is the solution of \( x - 4 = \sqrt{2x} \)?

3. What is \((-27)^{\frac{1}{3}}\) in simplified form?
4. Given \( f(x) = \sqrt{x - 1} + 2 \) and \( g(x) = \sqrt{x} \).

Part A: Describe how to obtain the graph of \( f(x) \) from the graph of \( g(x) \).

Answer and Explain

a horizontal shift 1 unit to the right and a vertical shift of 2 units up

Part B: State the domain and range of \( f(x) \).

Answer and Explain

domain: \( x \geq 1 \); range: \( f(x) \geq 2 \)

Part C: Graph \( f(x) \) and \( g(x) \) on the grid provided.