Radical and Rational Functions

Unit 5

Unit Overview

In this unit, you will extend your study of functions to radical, rational, and inverse functions and the composition of functions. You will solve rational equations and inequalities as well as equations with rational exponents.

Unit 5 Academic Vocabulary

Add these words and others that you encounter in this unit to your vocabulary notebook.

- complex fraction
- horizontal asymptote
- inverse variation
- one-to-one function
- power function
- rational exponent
- rational function
- vertical asymptote

Essential Questions

Why is it important to consider the domain and range of a function?

How are inverse functions useful in everyday life?

EMBEDDED ASSESSMENTS

This unit has two embedded assessments, following Activities 5.3 and 5.7. These assessments will allow you to demonstrate your understanding of inverse functions, the composition of functions, and solving and graphing radical and rational equations.

Embedded Assessment 1

Square Root Expressions, Equations and Functions p. 291

Embedded Assessment 2

Rational Equations and Functions p. 323

UNIT 5

Getting Ready

Write your answers on notebook paper or grid paper. Show your work.

1. What values are not possible for the variable x in each expression below? Explain your reasoning.

a. $\frac{2}{x}$ **b.** $\frac{2}{x-1}$

2. Perform the indicated operation.

a. $\frac{2x}{5} - \frac{3x}{10}$

b. $\frac{2x+1}{x+3} + \frac{4x-3}{x+3}$

c. $\frac{2}{x} + \frac{5}{x+1}$

3. Simplify each monomial.

a. $(2x^2y)(3xy^3)$

b. $(4ab^3)^2$

Factor each expression in Items 4–5.

4. $81x^2 - 25$

5. $2x^2 - 5x - 3$

6. Simplify $\sqrt{128x^3}$.

7. Find the composition $f(g(x))$ if $f(x) = 5x - 4$ and $g(x) = 2x$.

8. Which of the following is the inverse of $h(x) = 3x - 7$?

a. $7 - 3x$

b. $3x + 7$

c. $\frac{x+7}{3}$

d. $\frac{1}{3x-7}$

Composition of Functions
Code Breakers

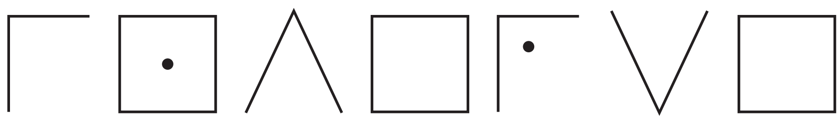
ACTIVITY
5.1

SUGGESTED LEARNING STRATEGIES: Shared Reading, Marking the Text, Summarize/Paraphrase/Retell, Look for a Pattern, Quickwrite

The use of *cryptography* goes back to ancient times. In ancient Greece, Spartan generals exchanged messages by wrapping them around a rod called a scytale and writing a message on the adjoining edges. The Roman general and statesman Julius Caesar used a transposition cipher that translated letters three places forward in the alphabet. For example, the word CAT was encoded as FDW.

In modern times, cryptography was used to secure electronic communications. Soon after Samuel F.B. Morse invented the telegraph in 1844, its users began to *encode* the messages with a secret code, so that only the intended recipient could *decode* them. During World War II, British and Polish cryptanalysts used computers to break the German Enigma code so that secret messages could be deciphered.

Many young children practice a form of cryptography when writing notes in secret codes. The message below is written in a secret code.



1. Try to decipher the seven-letter word coded above.

2. What do you need to decipher the seven-letter word?

My Notes

CONNECT TO SCIENCE

Cryptography is the science of code-making (encoding) and *cryptanalysis* is the science of code-breaking (decoding).

CONNECT TO COMPUTING

Modern computers have completely changed the science of cryptography. Before computers, cryptography was limited to two basic types: transposition, rearranging the letters in a message, and substitution, replacing one letter with another. The most sophisticated pre-computer codes used five or six operations. Computers can now use thousands of complex algebraic operations to encrypt messages.

My Notes

SUGGESTED LEARNING STRATEGIES: Look for a Pattern, Quickwrite, Create Representations, Marking the Text, Summarize/Paraphrase/Retell

3. The following message uses a numerical code. Can you decode the four-letter word? Explain how you know.

13 1 20 8

4. What is this six-letter word?

21 5 10 17 17 14

LETTER-TO-NUMBER CODES

A = 1	N = 14
B = 2	O = 15
C = 3	P = 16
D = 4	Q = 17
E = 5	R = 18
F = 6	S = 19
G = 7	T = 20
H = 8	U = 21
I = 9	V = 22
J = 10	W = 23
K = 11	X = 24
L = 12	Y = 25
M = 13	Z = 26

In Item 3, a single function was used to encode a word. The function assigned each letter to the number representing its position in the alphabet.

In Item 4, two functions were used to encode a word. The first function assigned each letter to the number representing its position in the alphabet x and then the function $f(x) = x + 2$ was used to *encode* the message further as shown in the table.

LETTER	x	$f(x)$
S	19	21
C	3	5
H	8	10
O	15	17
O	15	17
L	12	14

SUGGESTED LEARNING STRATEGIES: Create Representations, Look for a Pattern, Work Backward

My Notes

5. Write a function g that could *decode* the message in Item 4 and use it to complete the table below.

$g(x) =$ ______________

x	$g(x)$	LETTER
21		S
5		C
10		H
17		O
17		O
14		L

6. Try to decipher the more difficult message below. First, each letter in the message was assigned a number based on its position in the alphabet, and then another function encoded the message further.

20 -1 50 8 11 50

7. The encoding function for Item 6 is $f(x) =$ ______________.

Write a decoding function g and complete the table below.

$g(x) =$ ______________

x	$g(x)$	LETTER
20		
-1		
50		
8		
11		
50		

SUGGESTED LEARNING STRATEGIES: Guess and Check, Interactive Word Wall, Vocabulary Organizer, Create Representations, Simplify the Problem, Think/Pair/Share

My Notes

MATH TIP

$f(g(x))$ indicates a **composition of functions,** in which the range of function g becomes the domain for function f.

CONNECT TO AP

The notion of inverse is a very important concept in the study of functions in advanced mathematics courses.

READING MATH

The symbol for the inverse of function f is f^{-1}. You read this symbol as "f inverse."

8. Verify that your function works by encoding the letter $C = 3$ with f and then decoding it by using g.

Recall that functions f and g are called **inverse functions** if and only if $f(g(x)) = x$ for all x in the domain of g and $g(f(x)) = x$ for all x in the domain of f.

9. Use the definition of inverse functions to show that the encoding function $f(x) = 3x - 4$ and its decoding function g are inverses.

10. What is f^{-1} for the function $f(x) = x + 2$?

11. What is f^{-1} for the function $f(x) = 3x - 4$?

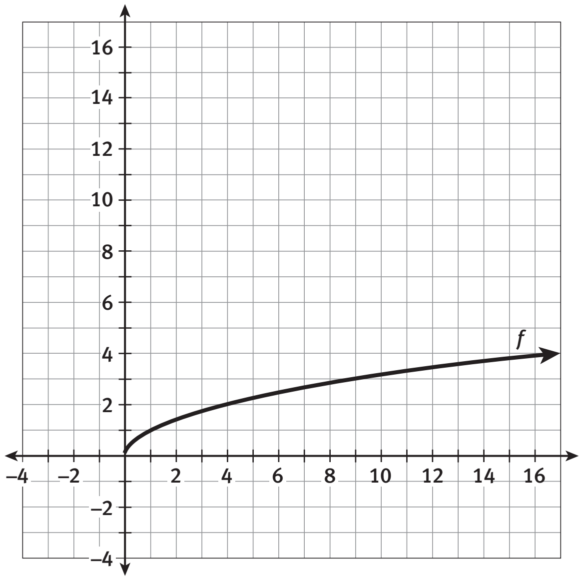
SUGGESTED LEARNING STRATEGIES: Create Representations, Quickwrite, Look for a Pattern, Group Presentation

My Notes

So far, the functions in this activity have been linear functions. Other types of functions also have inverses.

12. The graph of $f(x) = \sqrt{x}$ is shown.

a. List four points on the graph of f and four points on its inverse.



b. Use the points from Part (a) to graph the inverse of f.

c. Find f^{-1} algebraically.

d. Graph f and f^{-1} on a calculator. Is f^{-1} on your calculator the same as the graph in Part (b)? Explain below.

e. What are the domain and range of f?

f. Based on your answer in Part (b), what should be the domain and range of f^{-1}?

g. Use your results from Part (f) to complete the following.

$f^{-1}(x) =$ ______________ for $x \geq$ _____

MATH TIP

Recall that if (a, b) was a point on the graph of a function then (b, a) must be a point on the graph of the inverse of the function.

SUGGESTED LEARNING STRATEGIES: Interactive Word Wall, Vocabulary Organizer, Create Representations, Quickwrite, Look for a Pattern, Work Backward

My Notes

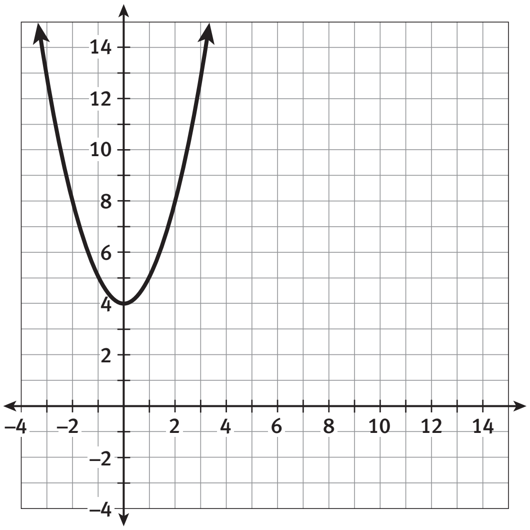
MATH TERMS

A **relation** is a set of ordered pairs that may or may not be defined by a rule. Not all relations are functions, but all functions are relations.

All functions have an inverse *relation*, but the relation may or may not be a function.

13. Use the quadratic function g graphed below.

a. Graph the inverse of g.



b. Is the inverse of g a function? Explain your reasoning.

c. What characteristic of the graph of a function can you use to determine whether its inverse relation is a function?

d. The quadratic function shown in the graph is $g(x) = x^2 + 4$. Find an equation for the inverse relation of this function.

SUGGESTED LEARNING STRATEGIES: Interactive Word Wall, Vocabulary Organizer, Quickwrite, Create Representations

A function is defined as **one-to-one** if, for each number in the range of the function, there is exactly one corresponding number in the domain of the function.

14. Is g from Item 13 a one-to-one function? Explain.

15. What do you know about a function whose inverse relation is a function?

16. Determine whether each type of function will *always* have an inverse that is a function. Explain your reasoning.

a. Linear function

b. Quadratic function

17. Investigate the cubic function $h(x) = x^3 - 6x^2 + 8x + 5$.

a. Use your calculator to graph h in the viewing window $[-10, 16]$ by $[-10, 16]$ and sketch the results in the My Notes section.

b. Explain whether or not h is a one-to-one function.

My Notes

ACADEMIC VOCABULARY

A one-to-one function is a function in which for each element of the range, there is exactly one corresponding element in the domain.

SUGGESTED LEARNING STRATEGIES: Look for a Pattern, Create Representations, Quickwrite

My Notes

17. (*continued*)

c. Is your answer in Part (b) true for all cubic functions? Explain.

d. Does a cubic function always have an inverse that is a function? Explain your reasoning.

Three students are encoding and decoding messages that begin with the numerical code used in Item 3. Suppose that the message HELLO is to be passed from one student to a second student and then on to a third student. Use this information for Items 18–24.

18. The first student translates HELLO to numbers and then encodes it with the function $f(x) = -2x + 12$. What encoded message will the second student receive?

19. After receiving the encoded message, the second student encodes it again, using the function $g(x) = -x + 9$. What encoded message will the third student receive?

20. Let h be the composite function $h(x) = g(f(x))$.

a. Write a rule for h.

b. Explain how h relates to the encoding of the message HELLO.

SUGGESTED LEARNING STRATEGIES: Work Backward, Create Representations, Quickwrite, Summarize/Paraphrase/Retell, Look for a Pattern

My Notes

21. The third student receives another message shown below. What does this message say?

27 25 7   37 27   27 25 7

22. Suppose that the message above had been encoded using the composite function $k(x) = f(g(x))$. How would the message have been encoded?

23. Can the rule found in Item 20(a) also be used for k? Explain.

24. Suppose that the third student is given f and g, and the encoded message. What additional information would help this student to decode the message more efficiently? Explain.

25. Another message is encoded twice. First, the message is encoded with the function $f(x) = 2x - 29$, and then it is encoded with the function $g(x) = x^2$.

a. Encode the word GRAPH using the composite function.

b. Decode the following message.

1 1 121   1 1 361   121 1   1 1 361

SUGGESTED LEARNING STRATEGIES: Think/Pair/Share, Quickwrite

My Notes

26. Consider $h(x) = g(f(x))$.

a. Is h a function?

b. Is h a one-to-one function?

c. How do your answers in Parts (a) and (b) relate to your work in Item 25? Explain.

CHECK YOUR UNDERSTANDING

Write your answers on notebook paper. Show your work.

1. Find the inverse of each function. State the domain and range of the function and its inverse.

   a. $f(x) = x^3 - 6$

   b. $f(x) = 2\sqrt{x - 5}$

2. Which functions have inverse functions? Explain your reasoning.

   a. $f(x) = 10^x$ **b.** $f(x) = x^2 - 10$

   c.

x	-3	-1	0	2
$f(x)$	-1	0	-1	3

3. Given $f(x) = 2x - 3$ and $f(x) = x^2 - 2x - 8$.

   a. Find $y = g(f(x))$.

   b. Is the inverse of y a function? Explain.

4. Classify each statement as *always, sometimes,* or *never* true. Explain your reasoning.

   a. The inverse of a linear function is also a function.

   b. The inverse of a quadratic function is also a function.

   c. The inverse of a cubic function is also a function.

5. **MATHEMATICAL REFLECTION** How does the graph of a quadratic function demonstrate whether its inverse will be a function or not?

Graphing Square Root Functions
Go, Boat, Go!

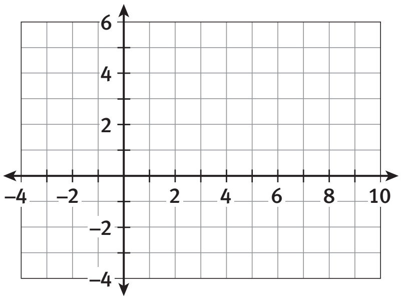
ACTIVITY
5.2

SUGGESTED LEARNING STRATEGIES: Activating Prior Knowledge, Create Representations, Quickwrite

Suppose the hull speed in knots H of a sailboat is given by the function $H(x) = 1.34\sqrt{x}$, where x is the length of the boat in feet at the waterline.

1. The hull speed function is a transformation of the parent square root function $f(x) = \sqrt{x}$.

a. Graph H and f on the same axes. How do these graphs compare to each other?



b. What are the domain and the range of f?

c. What are the domain and range of H?

2. Explain how you could use transformations of the graph of $f(x) = \sqrt{x}$ to graph $g(x) = 2\sqrt{x}$.

My Notes

CONNECT TO SAILING

The speed of a boat is measured in knots (nautical miles per hour). The distance it travels in water is measured in nautical miles. A nautical mile is equal to 1.15 statute miles.

MATH TIP

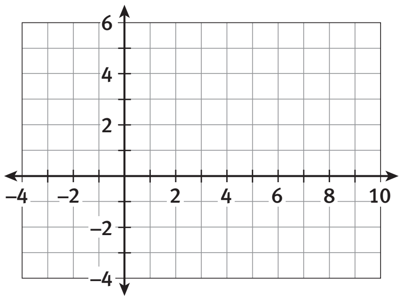
To graph the parent square root function, use key points with x-values that are perfect squares, such as 0, 1, 4, and 9.

SUGGESTED LEARNING STRATEGIES: Quickwrite, Create Representations, Think/Pair/Share, Look for a Pattern

My Notes

3. How does the graph of $g(x) = \sqrt{x - 3}$ compare to the graph of $f(x) = \sqrt{x}$?

4. Sketch g and f from Item 3 on the same axes below.



5. What is the domain and range of g?

Multiple transformations can be applied to the basic function to create a new function.

6. Describe the transformations of $f(x) = \sqrt{x}$ that result in the functions listed below.

a. $g(x) = -\sqrt{x + 2}$

b. $h(x) = \sqrt{x - 3} + 4$

MATH TIP

Recall that the function $y = c \cdot f(x)$ represents a *vertical stretch or shrink* of the original function $y = f(x)$ after the y-values have been multiplied by c.

MATH TIP

Recall that the function $y = -f(x)$ represents a *reflection over the x-axis* of the original function $y = f(x)$ after the y-values have been multiplied by -1.

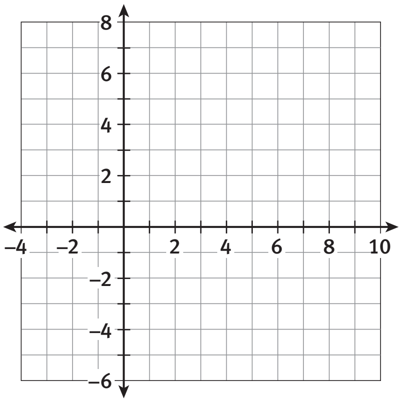
MATH TIP

Recall that the function $y = f(x + c)$ results in a *horizontal translation* of the original function $y = f(x)$ and $y = f(x) + c$ results in a *vertical translation* of the original function.

SUGGESTED LEARNING STRATEGIES: Predict and Confirm, Create Representations, Group Presentation

My Notes

7. Sketch the graph of each function in Item 6. Then state the domain and range for each function. Use a calculator to check your results.



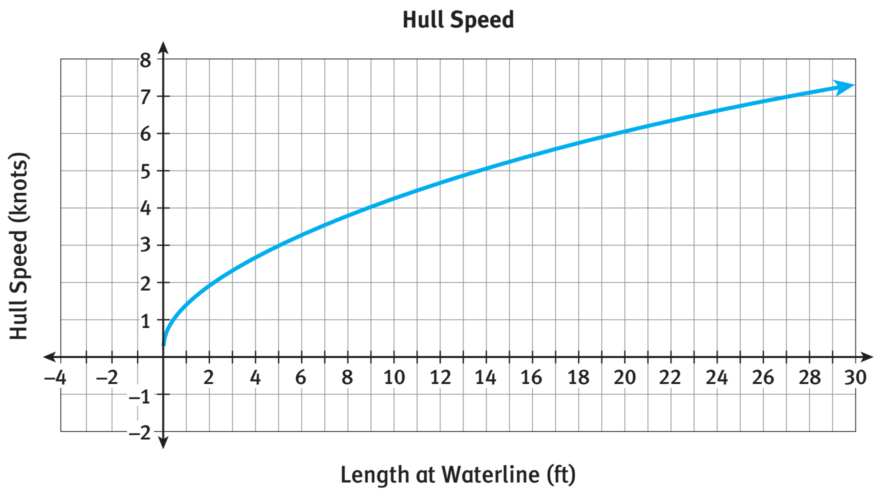
8. Without graphing, determine the domain and range of $f(x) = \sqrt{x + 5} - 1$.

Graphing Square Root Functions
Go, Boat, Go!

SUGGESTED LEARNING STRATEGIES: Work Backward, Guess and Check, Create Representations, Think/Pair/Share, Note Taking, Interactive Word Wall, Vocabulary Organizer

My Notes

The graph of the hull speed of a sailboat *H* is shown below.



9. Use the graph to estimate the hull speed of a sailboat that is 24 ft long at the waterline.

10. Use the graph to estimate the length at the waterline of a sailboat whose hull speed is 6 knots.

11. Write an equation that could be used to determine the length at the waterline of a sailboat with a hull speed of 6 knots.

MATH TIP

An **extraneous solution** can be introduced when you square both sides of an equation to eliminate the square root. The resulting equation may not be equivalent to the original for all values of the variable.

To solve **square root equations**, follow these steps.

Step 1: *Isolate the radical term.*

Step 2: *Square both sides of the equation.*

Step 3: *Solve for the unknown(s).*

Step 4: *Check for* ***extraneous solutions.***

SUGGESTED LEARNING STRATEGIES: Note Taking, Identify a Subtask, Guess and Check, Group Presentation

My Notes

EXAMPLE 1

Solve the equation $\sqrt{x-3}+4=9$.

		$\sqrt{x-3}+4=9$
Step 1:	*Isolate the radical.*	$\sqrt{x-3}=5$
Step 2:	*Square both sides.*	$(\sqrt{x-3})^2=(5)^2$
Step 3:	*Solve the equation.*	$x-3=25$
		$x=28$

Check the solution.

Step 4:	*Substitute 28 into the original equation.*	$\sqrt{28-3}+4 \mathrel{?} 9$
		$5+4=9$

EXAMPLE 2

Solve the equation $x=\sqrt{x+1}+5$.

Step 1:	*Isolate the radical.*	$x=\sqrt{x+1}+5$
		$x-5=\sqrt{x+1}$
Step 2:	*Square both sides.*	$(x-5)^2=(\sqrt{x+1})^2$
		$x^2-10x+25=x+1$
Step 3:	*Solve for x.*	$x^2-11x+24=0$
		$(x-3)(x-8)=0$
	possible solutions	$x=3, 8$

Check the possible solutions.

Step 4:	$3 \mathrel{?} \sqrt{3+1}+5$	$8 \mathrel{?} \sqrt{8+1}+5$
	$3 \neq 2+5$	$8=3+5$

Only $x=8$ is a solution.

TRY THESE A

Solve each equation.

a. $2-\sqrt{x+1}=-5$

b. $\sqrt{x+4}=x-8$

SUGGESTED LEARNING STRATEGIES: Marking the Text, Predict and Confirm, Create Representations, Quickwrite

My Notes

12. Solve the hull speed equation you wrote in Item 11.

13. Maggie claims that her 27-foot sailboat *My Hero* has a hull speed of 7 knots. The length of her boat at the waterline is 24 ft. Is this claim reasonable? Explain why or why not.

CHECK YOUR UNDERSTANDING

Write your answers on notebook paper or grid paper. Show your work.

Describe each function as a transformation of $f(x) = \sqrt{x}$. State the domain and range.

1. $f(x) = 2\sqrt{x - 3}$
2. $f(x) = 4 - \sqrt{x + 1}$
3. $f(x) = 3\sqrt{x - 5} - 1$

Graph each function, using your knowledge of transformations.

4. $f(x) = \sqrt{x + 1} - 3$
5. $f(x) = -3\sqrt{x} + 1$
6. $f(x) = 1 - \sqrt{x - 2}$

Solve for x.

7. $\sqrt{x - 1} = 4$
8. $x + \sqrt{x - 1} = 7$
9. $2 + \sqrt{x} = 12$
10. $\sqrt{x + 4} + 7 = 3$
11. **MATHEMATICAL REFLECTION** What have you learned about graphing radical functions and solving radical equations in this activity?

Rational Exponents and Radical Expressions
A Mighty Wind

ACTIVITY 5.3

SUGGESTED LEARNING STRATEGIES: Shared Reading, Summarize/Paraphrase/Retell, Marking the Text, Note Taking, Think/Pair/Share

In 1805, Sir Francis Beaufort, a British admiral, devised a 13-point scale for measuring wind force based on how a ship's sails move in the wind. The scale is widely used by sailors even today to interpret the weather at sea and has also been adapted for use on land. It was not until the early 1900s that the Beaufort scale was related to an average wind velocity v in miles per hour using the equation $v = 1.87B^{\frac{3}{2}}$ where B is the Beaufort scale number.

1. The weather service issued a small-craft advisory with force 6 winds. Determine the wind speed predicted by the equation above, using a calculator.

The definition below relates a **rational exponent** to a radical expression. You can use this definition to evaluate expressions without a calculator.

Definition of Rational Exponents

$$a^{\frac{1}{n}} = \sqrt[n]{a} \text{ for a} > 0 \text{ and integer } n$$
$$a^{\frac{m}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m \text{ for } a > 0 \text{ and integers } m \text{ and } n$$

2. Without using a calculator, find the wind velocity predicted by the equation $v = 1.87B^{\frac{3}{2}}$ for gale force winds ($B = 9$). Verify your answer, using a calculator.

The definition of rational exponents is also useful for simplifying expressions.

EXAMPLE 1

Simplify each expression, using the definition of rational exponents.

a. $81^{\frac{1}{2}} = \sqrt{81} = 9$

b. $9^{\frac{3}{2}} = (\sqrt{9})^3 = 3^3 = 27$

c. $16^{\frac{3}{4}} = \sqrt[4]{16}^3 = 2^3 = 8$

TRY THESE A

Simplify each expression, using the definition of rational exponents. Write your answers on a separate sheet of paper. Show your work.

a. $81^{\frac{3}{4}}$ **b.** $27^{\frac{2}{3}}$ **c.** $100^{\frac{5}{2}}$

My Notes

CONNECT TO METEOROLOGY

The Beaufort wind scale for use at sea is shown below.

Force	Description
0	calm
1–3	light/gentle breeze
4–5	moderate/brisk breeze
6–7	strong winds, small-craft advisory
8–9	gale force winds
10–11	storm force winds, wind damage
12	hurricane force winds

ACADEMIC VOCABULARY

rational exponent

MATH TIP

$\sqrt[n]{a}$ is called the n^{th} root of the radicand a. For square roots, the index n is not written.

The positive square root of 49 is $\sqrt{49} = 7$.

The cube root of 8 is $\sqrt[3]{8} = 2$.

My Notes

SUGGESTED LEARNING STRATEGIES: Think/Pair/Share, Simplify the Problem, Note Taking, Discussion Group, Group Presentation

The rules of exponents that you learned in previous mathematics classes apply to both rational and real number exponents.

EXAMPLE 2

Simplify each expression, using the properties of exponents.

a. $4^{\frac{5}{2}} \cdot 4^{\frac{1}{2}} = 4^{\frac{6}{2}} = 4^3 = 64$

b. $\left(3^{\frac{1}{3}}\right)^4 = 3^{\frac{4}{3}} = \sqrt[3]{3^4} = \sqrt[3]{3^3 \cdot 3} = 3\sqrt[3]{3}$

c. $\dfrac{8^{\frac{1}{2}}}{2^{\frac{1}{2}}} = \left(\dfrac{8}{2}\right)^{\frac{1}{2}} = 4^{\frac{1}{2}} = 2$

d. $(2^{\pi})^2 = 2^{2\pi} \approx 2^{6.28} \approx 77.7$

TRY THESE B

Simplify each expression using the properties of exponents. Write your answers in the My Notes space. Show your work.

a. $(9^3)^{\frac{1}{2}}$

b. $18^{\frac{3}{2}} \cdot 18$

c. $3^{\sqrt{2}} \cdot 2^{\sqrt{2}}$

MATH TIP

The *rules of exponents* given $a > 0$, $b > 0$, and real numbers m and n are listed below.

$a^m \cdot a^n = a^{m+n}$

$\dfrac{a^m}{a^n} = a^{m-n}$

$(a^m)^n = a^{m \cdot n}$

$\dfrac{1}{a^m} = a^{-m}$

$a^m \cdot b^m = (ab)^m$

$\dfrac{a^m}{b^m} = \left(\dfrac{a}{b}\right)^m$

Variable expressions can also be simplified by using the properties of exponents.

EXAMPLE 3

Simplify $x^{\frac{3}{2}} \cdot y^{\frac{5}{2}}$ and write in radical form. Assume $x > 0$ and $y > 0$.

	Original expression	$x^{\frac{3}{2}} \cdot y^{\frac{5}{2}}$
Step 1:	*Write using definition of rational exponents.*	$\sqrt{x^3} \cdot \sqrt{y^5}$
Step 2:	*Multiply like radicals.*	$\sqrt{x^3y^5}$
Step 3:	*Factor perfect squares.*	$\sqrt{x^2y^4\,xy}$
Step 4:	*Simplify perfect squares.*	$xy^2\sqrt{xy}$

TRY THESE C

Simplify each expression and write in radical form. Assume all variables are greater than 0.

a. $\dfrac{x^{\frac{4}{3}} \cdot y^{\frac{5}{3}}}{x^{\frac{1}{3}}}$

b. $\left(x^{\frac{1}{3}}\right)^{13}$

SUGGESTED LEARNING STRATEGIES: Activating Prior Knowledge, Think/Pair/Share, Note Taking, Discussion Group, Quickwrite, Marking the Text, Summarize/Paraphrase/Retell, Create Representations, Look for a Pattern

My Notes

You can also solve equations by using the properties of exponents.

EXAMPLE 4

Solve the equation $x^{\frac{2}{3}} = 36$. Assume $x > 0$.

Original equation $\qquad x^{\frac{2}{3}} = 36$

Step 1: *Raise both sides to a power that makes the exponent on x equal to 1* $\qquad \left(x^{\frac{2}{3}}\right)^{\frac{3}{2}} = 36^{\frac{3}{2}}$

Step 2: *Simplify using the definition of rational exponents.*

$$x = \left(\sqrt{36}\right)^3$$
$$x = 6^3$$
$$x = 216$$

TRY THESE D

Solve each equation. Assume $x > 0$.

a. $x^{\frac{1}{4}} = 3$

b. $x^{\frac{5}{2}} = 32$

c. $2x^3 = 54$

MATH TIP

Use the Power of a Power Property to isolate the variable so that it has an exponent of 1.

Example: $(x^5)^{\frac{1}{5}} = x$

3. If x could be any real number, not just a positive one, would there be any other solutions to the equation shown in Example 4 above?

CONNECT TO AP

In AP Calculus, you will work extensively with fractional and negative exponents.

4. Suppose the wind speed is measured at 30 mph.

a. Write an equation that could be solved to find the Beaufort number associated with this wind speed.

b. Solve the equation for B.

c. Use the table on the first page of the activity to interpret the wind speed.

SUGGESTED LEARNING STRATEGIES: Activating Prior Knowledge, Note Taking, Look for a Pattern, Create Representations, Think/Pair/Share, Interactive Word Wall, Vocabulary Organizer

My Notes

The properties of algebra that you used to add, subtract, and multiply polynomials extend to radical expressions.

Multiply Two Binomials

$$\begin{aligned}(x-3)(x+2) &= x^2 + 2x - 3x - 6 \\ &= x^2 - x - 6\end{aligned}$$

5. Use the problem above as a model to multiply $(\sqrt{x} - 3)(\sqrt{x} + 2)$.

MATH TIP

Difference of two squares
$(a + b)(a - b) = a^2 - b^2$

Perfect square trinomials
$(a \pm b)^2 = a^2 \pm 2ab + b^2$

TRY THESE E

Simplify each expression.

a. $(\sqrt{x} + 5)(\sqrt{x} - 3)$

b. $(3 + \sqrt{5})^2$

c. $(\sqrt{3} + 1)(\sqrt{3} - 1)$

Radical expressions are often written in simplest form.

A radical expression with nth roots is in **simplest form** when:
- the radicand contains no perfect n^{th} power factors other than 1
- there are no fractions in the radicand
- there are no radicals in the denominator of a fraction

The denominator of any radical expression can be rationalized by multiplying by the number 1.

EXAMPLE 5

Simplify each expression.

a. $\dfrac{2}{\sqrt{5}} = \dfrac{2}{\sqrt{5}} \cdot \dfrac{\sqrt{5}}{\sqrt{5}} = \dfrac{2\sqrt{5}}{(\sqrt{5})^2} = \dfrac{2\sqrt{5}}{5}$

b. $\sqrt[3]{\dfrac{8}{5}} = \dfrac{\sqrt[3]{8}}{\sqrt[3]{5}} = \dfrac{2}{\sqrt[3]{5}} \cdot \dfrac{\sqrt[3]{5^2}}{\sqrt[3]{5^2}} = \dfrac{2\sqrt[3]{25}}{\sqrt[3]{5^3}} = \dfrac{2\sqrt[3]{25}}{5}$

c. $\dfrac{3}{1+\sqrt{5}} = \dfrac{3}{1+\sqrt{5}} \cdot \dfrac{1-\sqrt{5}}{1-\sqrt{5}} = \dfrac{3(1-\sqrt{5})}{1-\sqrt{5}+\sqrt{5}-(\sqrt{5})^2}$

$= \dfrac{3-3\sqrt{5}}{1-5} = \dfrac{3-3\sqrt{5}}{-4} = -\dfrac{3}{4} + \dfrac{3\sqrt{5}}{4}$

MATH TIP

The binomial expressions $a + b$ and $a - b$ are **conjugates** of each other.

6. In each example above, circle the factor equal to the number 1.

SUGGESTED LEARNING STRATEGIES: Simplify the Problem, Group Presentation, Interactive Word Wall, Vocabulary Organizer, Activating Prior Knowledge, Think/Pair/Share, Summarize/Paraphrase/Retell, Create Representations, Look for a Pattern, Quickwrite

My Notes

TRY THESE F

Simplify each expression. Write your answers in the My Notes space. Show your work.

a. $\frac{5}{\sqrt[3]{10}}$ **b.** $\frac{1+\sqrt{2}}{\sqrt{3}}$ **c.** $\frac{4}{\sqrt{5}-2}$ **d.** $\frac{2+\sqrt{3}}{3+\sqrt{7}}$

In a previous activity, you explored the graphs of square root functions. The square root function is an example of a *power function*. A function of the form $f(x) = x^n$ for any real number n is a **power function**.

ACADEMIC VOCABULARY

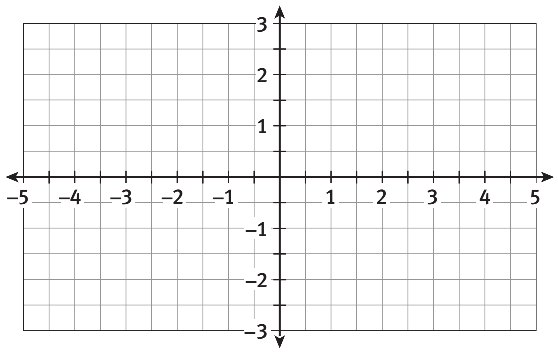
power function

7. Write the parent square root function as a power function.

8. List some other power functions that you have studied previously.

Other power functions with rational exponents have some interesting properties.

9. Graph the parent square root function and the power functions $f(x) = x^{\frac{1}{3}}$ and $f(x) = x^{\frac{1}{4}}$ on a graphing calculator. Then sketch the result on the grid below.



10. How do the graphs of these functions compare?

My Notes

SUGGESTED LEARNING STRATEGIES: Quickwrite, Predict and Confirm, Look for a Pattern, Summarize/Paraphrase/Retell, Create Representations, Quickwrite

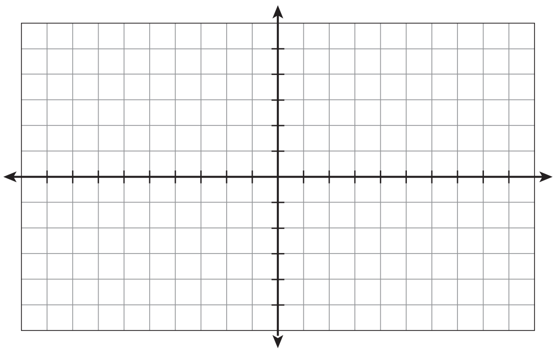
11. Why is the function $f(x) = x^{\frac{1}{3}}$ defined for all real values of x while the other two functions in Item 9 are not?

12. Without graphing, predict the domain of each function.

a. $f(x) = x^{\frac{1}{5}}$

b. $f(x) = x^{\frac{1}{6}}$

13. Graph the parent square root function and the power functions $f(x) = x^{\frac{3}{2}}, f(x) = x^{\frac{1}{3}}$, and $f(x) = x^{\frac{2}{3}}$ on a graphing calculator. Then sketch the result on the grid below.



14. How do the graphs of these functions compare?

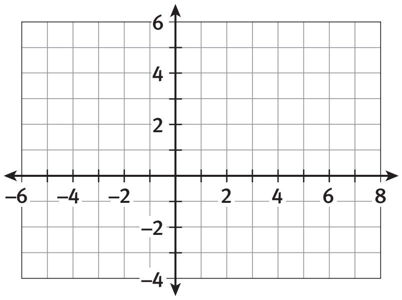
SUGGESTED LEARNING STRATEGIES: Create Representations, Look for a Pattern, Discussion Group, Quickwrite, Marking the Text, Activating Prior Knowledge, Note Taking, Group Presentation

My Notes

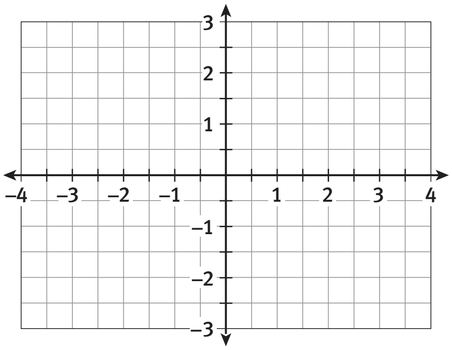
15. How do the graphs of the functions in Item 13 help to explain the fact that an equation like $x^{\frac{3}{2}} = 8$ will only have 1 solution but an equation like $x^{\frac{2}{3}} = 8$ will have 2 solutions?

16. Solve each equation and inequality, using a graphing calculator. Enter the left side as one function and the right side as another function. Solve graphically or algebraically.

a. $2\sqrt{x + 4} = 6$



b. $x^{\frac{2}{3}} = 2$



SUGGESTED LEARNING STRATEGIES: Marking the Text, Create Representations, Group Presentation

My Notes

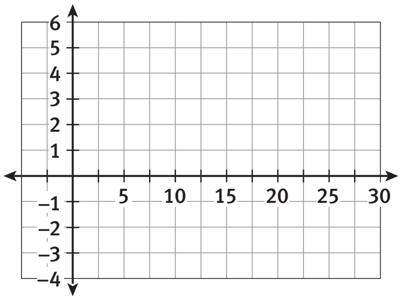
MATH TIP

If $f(x) > k$, the solution interval will be those x-values in the function domain where $f(x)$ is *above* the line $y = k$.

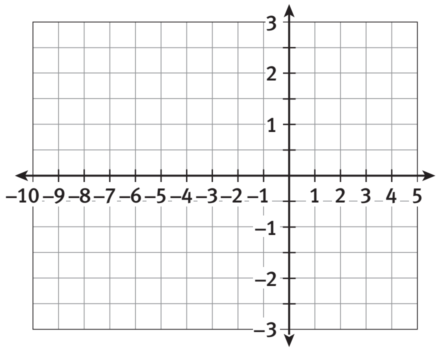
If $f(x) < k$, the solution interval will be those x-values in the function domain where $f(x)$ is *below* the line $y = k$.

16. (*continued*)

c. $\sqrt{x-1} > 5$



d. $\sqrt[3]{x-1} + 4 \leq 2$



CHECK YOUR UNDERSTANDING

Write your answers on notebook paper or grid paper. Show your work.

Simplify each expression.

1. $125^{\frac{2}{3}}$

2. $8^{\frac{3}{4}} \cdot 8^{\frac{1}{2}}$

3. $\left(\frac{27}{25}\right)^{-\frac{1}{2}}$

4. $\left(x^{\frac{3}{2}} y^{\frac{1}{2}}\right)^3$

5. $\sqrt{\sqrt[3]{x^8 y^6}}$

6. $\left(\frac{x^4}{y^6}\right)^{\frac{1}{3}}$

Solve each equation. Assume $x > 0$.

7. $5x^{\frac{1}{3}} = 50$

8. $4x^4 - 324 = 0$

Simplify each expression.

9. $(\sqrt{x} - 8)^2$

10. $\sqrt{x+y} \cdot \sqrt{x-y}$

11. $\dfrac{1 + \sqrt[3]{2}}{\sqrt[3]{2}}$

12. Describe $g(x) = 2x^{\frac{1}{3}} - 3$ as a transformation of $f(x) = x^{\frac{1}{3}}$. Use this information to sketch the graph of g.

13. Solve graphically or algebraically.

a. $\sqrt{x+2} - 3 > 4$

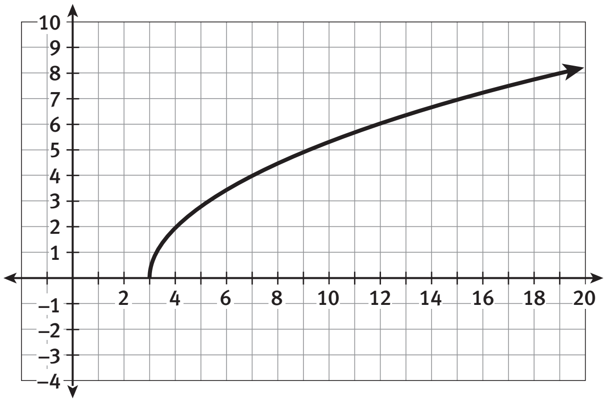
b. $3x^{\frac{2}{3}} = 12$

14. **MATHEMATICAL REFLECTION** How does the mathematics of this activity relate to what you have learned previously about simplifying expressions and solving equations?

Square Root Expressions, Equations, and Functions

A MIGHTIER WIND

1. The graph of a function g is shown below.



a. Describe the graph as a transformation of $f(x) = \sqrt{x}$.

b. Write the equation for g.

c. State the domain and range of g.

d. Find the inverse of g. Be sure to include any restrictions on the domain of the inverse.

e. Use the graph or a table to solve the inequality $g(x) > 7$.

2. Solve the equation $x + \sqrt{x} = 6$.

3. Classify each statement as *sometimes*, *always*, or *never true*.

a. $2^{\frac{2}{3}} \cdot 4^{\frac{4}{3}} = 8 \cdot 2^{\frac{1}{3}}$

b. If $x > 0$, then $(\sqrt{x} \cdot \sqrt{2})^3 = x^{\frac{3}{2}}\ 2^{\frac{3}{2}}$.

c. If $x > 0$, then $(\sqrt{x} + \sqrt{2})^3 = x^{\frac{3}{2}} + 2^{\frac{3}{2}}$.

d. The only solution to $x^{\frac{2}{3}} = 4$ is $x = 8$.

Square Root Expressions, Equations, and Functions

A MIGHTIER WIND

4. The International Tornado Intensity Scale (T-Scale) is widely used in Europe to describe the wind force of tornados. It was developed to extend the Beaufort scale to classify meteorological events with very high wind speeds. A TORRO force $T = 0$ is the same as the Beaufort force $B = 8$ and a TORRO force $T = 2$ is the same the Beaufort force $B = 12$.

a. Write T as a linear function of B, using the ordered pairs (8, 0) and (12, 2).

b. Use what you have learned about inverse functions to solve for B as a function of T.

c. What is the value and meaning of $T(B(10))$?

d. Use composition of functions and the equation for wind velocity $v = 1.87B^{\frac{3}{2}}$ to express v as a function of T.

	Exemplary	Proficient	Emerging
Math Knowledge #1c, e; 2, 4b, c	The student: • States both the correct domain and range of g. (1c) • Solves the inequality correctly. (1e) • Solves the equation correctly. (2) • Solves for B as a function of T correctly. (4b) • Finds the correct value for $T(B(10))$. (4c)	The student: • States either the correct domain or range, but not both. • Uses the correct method to solve the equation but makes a computational or mathematical error. • Uses the correct method to solve for the function but makes a computational or mathematical error.	The student: • States neither the correct domain nor range. • Does not solve the inequality correctly. • Does not use the correct method to solve the equation. • Does not use the correct method to solve for the function. • Does not find the correct value for $T(B(10))$.
Problem Solving #1d, 3a–d	The student: • Finds the correct inverse of g and states its correct domain. (1d) • Classifies the four statements correctly. (3a–d)	The student: • Finds the correct inverse of g, but states an incorrect domain. OR • States the correct domain for an incorrect inverse. • Classifies only three of the four statements correctly.	The student: • Finds an incorrect inverse and an incorrect domain. • Classifies at least one of the statements correctly.
Representations #1b, 4a, d	The student: • Writes a correct equation for g. (1b) • Writes a correct linear function. (4a) • Correctly expresses v as a function of T. (4d)	The student: • Writes a partially correct equation for g. • Writes an incorrect linear function. • Writes a partially correct function.	The student: • Writes an incorrect equation for g. • Writes a function that is not linear. • Writes an incorrect function.
Communication #1a, 4c	The student: • Gives a complete and accurate description of the transformation. (1a) • Gives the correct meaning of $T(B(10))$. (4c)	The student: • Gives an incorrect description of the transformation with no mathematical errors. • Gives a partially correct meaning of $T(B(10))$.	The student: • Gives a description of the transformation that contains mathematical errors. • Gives an incorrect meaning of $T(B(10))$.

Introduction to Rational Functions
Planning a Summer Camp

ACTIVITY
5.4

SUGGESTED LEARNING STRATEGIES: Marking the Text, Summarize/Paraphrase/Retell, Create Representations, Look for a Pattern

My Notes

The finance committee of a nonprofit summer camp for children is setting the cost for a 5-day camp. The fixed cost for the camp is \$2400 per day, and includes things such as rent, salaries, insurance, and equipment. An outside food services company will provide meals at a cost of \$3 per camper, per meal. Campers will eat 3 meals a day.

As a nonprofit camp, the camp must cover its costs, but not make any profit. The committee must come up with a proposal for setting the fee for each camper, based on the number of campers who are expected to attend each week.

1. Initially, the committee decides to calculate camper fees based on the fixed cost of the camp alone, without meals for the campers.

a. What is the total fixed cost for the five days?

b. Complete the table below to determine the fee per camper that will guarantee the camp does not lose money.

Number of Campers	Fee per Camper
25	
50	
75	
100	
200	
500	
1000	
x	

MATH TIP

Use the patterns you observe in the table to write an algebraic expression in the last row when there are x campers.

SUGGESTED LEARNING STRATEGIES: Create Representations, Quickwrite, Group Presentation, Marking the Text

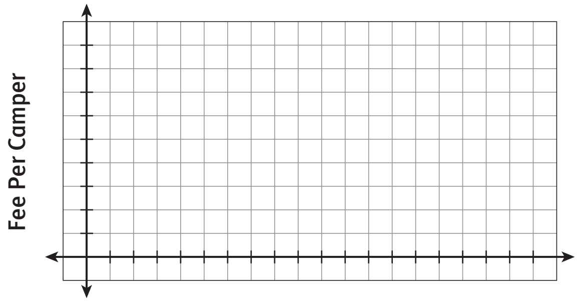
My Notes

MATH TIP

You can use the values in your table to help you determine an appropriate scale for a graph.

1. *(continued)*

c. Using an appropriate scale, make a graph showing the relationship between the fee per camper and the number of campers in attendance.



d. Write an algebraic rule for the fee per camper as a function of the number of campers in attendance.

2. Describe the features of the graph in Item 1(c).

3. Based on your work so far, is there a minimum camper fee, not counting the cost of meals? If so, what is it? Explain.

4. The function developed in Item 1 did not account for meals. Campers eat three meals per day at a cost of \$3 per camper per meal. The committee must determine a function that includes the cost of meals when setting the fee per camper.

a. What will be the total cost for meals per camper each week?

SUGGESTED LEARNING STRATEGIES: Create Representations, Look for a Pattern

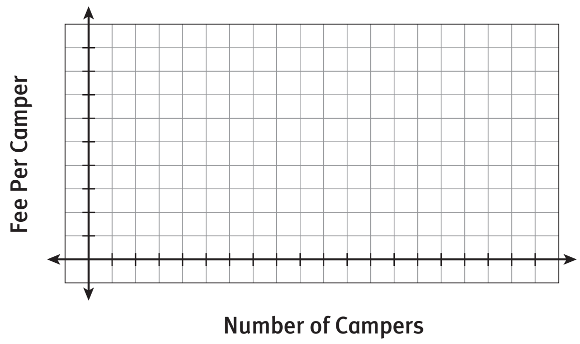
My Notes

4. (*continued*)

b. Complete the table below to determine the fee per camper that will guarantee the camp does not lose money.

Number of Campers	Fixed Cost plus the Cost of Meals	Fee per Camper
25		
50		
75		
100		
200		
500		
1000		
x		

c. Using an appropriate scale, make a graph showing the relationship between the fee per camper, including meals, and the number of campers.



d. Write an algebraic rule for the fee per camper, including meals, as a function of the number of children in attendance.

CONNECT TO AP

Describing the behavior of rational functions as they approach horizontal and vertical asymptotes provides an introduction to a more formal study of limits that will occur in calculus.

SUGGESTED LEARNING STRATEGIES: Quickwrite, Self Revision/Peer Revision, Group Presentation, Create Representations

My Notes

5. Based on your work so far, is there a minimum camper's fee? If so, what is it? Explain your reasoning.

6. How does your answer to Item 5 differ from the one you gave for Item 3?

7. Describe the difference between the graphs in Items 1(c) and 4(c).

8. The committee decides to award 30 scholarships to students who otherwise could not afford the camp. These scholarships include full use of the facilities and all meals at no charge.

a. To help account for the scholarships, complete the table below.

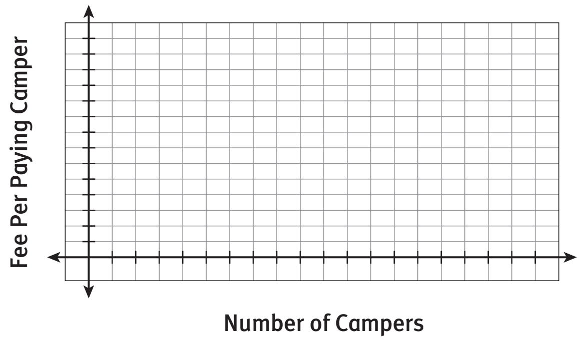
Number of Campers	Fixed Cost plus Cost of Meals	Number of Paying Campers	Fee per Paying Camper
50			
75			
100			
200			
500			
1000			
x			

SUGGESTED LEARNING STRATEGIES: Create Representations, Quickwrite, Self Revision/Peer Revision, Group Presentation

My Notes

8. (*continued*)

b. Using an appropriate scale, make a graph showing the relationship between the fee per paying camper and the number of campers.



c. Write an algebraic model for the fee per paying camper as a function of the number of campers in attendance.

9. Based on your work so far, is there a minimum camper's fee? If so, what is it? Explain.

10. How does your answer to Item 9 differ from the one you gave for Item 5?

11. How does your graph in Item 8(b) compare to the one in Item 4(c)?

SUGGESTED LEARNING STRATEGIES: Guess and Check, Work Backward, Quickwrite, Prewriting, RAFT

My Notes

12. If the number of campers is 25, what is the fee per paying camper? What does your answer tell you about the limitations of this model?

13. What is the domain of the function for the fee per paying camper?

14. Last year the weekly camper fee was $80. If the camp charges the same amount and grants 30 scholarships, what is the minimum number of paying campers that must attend so the camp does not lose money?

15. Express the number of campers as a function of the fee for each paying camper.

16. What is the relationship between the function in Item 8(c) and the function in Item 15?

The camp can accommodate up to 300 campers, and market research indicates that campers do not want to pay more than $200 per week. Although the camp is nonprofit, it cannot afford to lose money.

17. On a separate sheet of paper, write a proposal for setting the fee per camper. Be sure to include these items.

- the proposed fee
- the minimum number of campers needed to break even
- the maximum possible income for the proposed fee
- mathematics to support your reasoning

SUGGESTED LEARNING STRATEGIES: Vocabulary Organizer, Interactive Word Wall, Create Representations, Note Taking

When using a function to model a situation like the fee per camper, you only use those values that make sense in the context of the situation. Items 18–20 consider the **rational function** $f(x) = \frac{12000 + 45x}{x - 30}$ over a broader range of values.

18. Graph the function on a graphing calculator, using the viewing window $[-450, 450]$ by $[-400, 400]$.

a. Use your calculator to approximate the x- and y-intercepts.

b. Find the exact values of the x- and y-intercepts, using the function. Show your work.

c. Name the value(s) for which the function is not defined and explain how you determined the value(s). Recall that division of a nonzero quantity by zero is *undefined*.

d. What is the domain of the function?

e. What is the range of the function?

If the values of function f approach some number a as the absolute value of x becomes large without bound, the line $y = a$ is called a **horizontal asymptote** of f. If the absolute value of function f increases without bound as x approaches some number b, then the line $x = b$ is a **vertical asymptote** of f.

My Notes

ACADEMIC VOCABULARY

A **rational function** is a function that is the quotient of two polynomials. Its parent function is $f(x) = \frac{1}{x}$.

TECHNOLOGY TIP

Throughout this book, graphing calculator viewing window dimensions are given as [xmin, xmax] by [ymin, ymax].

ACADEMIC VOCABULARY

A horizontal asymptote is the line $y = a$ when the end behavior of a function approaches some constant a.

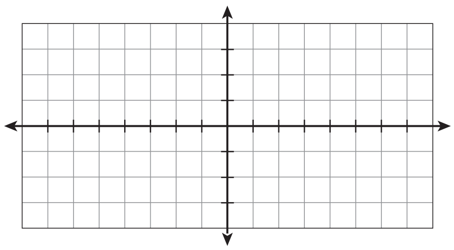
A vertical asymptote is the line $x = b$ if the absolute value of a function increases without bound as x approaches some number b.

SUGGESTED LEARNING STRATEGIES: Think/Pair/Share, Look for a Pattern, Create Representations, Group Presentation

My Notes

19. What are the vertical and horizontal asymptotes of $f(x) = \frac{12000 + 45x}{x - 30}$?

20. Sketch the graph of the function in Item 19 on the axes below. Indicate the scale, label the intercepts, and include the horizontal and vertical asymptotes.



CHECK YOUR UNDERSTANDING

Write your answers on notebook paper or grid paper. Show your work.

The population of grizzly bears in a remote area is modeled by the function $P(t) = \frac{200t - 120}{t + 0.5}$, where $t = 1$ represents the year 2001, $t = 2$ represents the year 2002, and so on.

1. Graph the grizzly bear population function.
2. Describe the features of the graph.
3. What is the domain and range of the function?
4. How many grizzly bears were there in 2005?
5. Predict the bear population in the year 2018.
6. Given the function $f(x) = \frac{x + 2}{x - 3}$,
   a. Identify any asymptotes of f.
   b. What are the x- and y-intercepts of f?
   c. Sketch the graph of f.
7. Given the function $f(x) = \frac{2x}{x - 5}$,
   a. Identify any asymptotes f.
   b. What are the x- and y-intercepts of f?
   c. Sketch the graph of f.
8. **MATHEMATICAL REFLECTION** What did you learn about rational functions while working on this activity?

Inverse Variation and Rational Functions
Stream Survival

ACTIVITY 5.5

SUGGESTED LEARNING STRATEGIES: Activating Prior Knowledge, Marking the Text, Quickwrite, Create Representations, Look for a Pattern, Group Presentation

My Notes

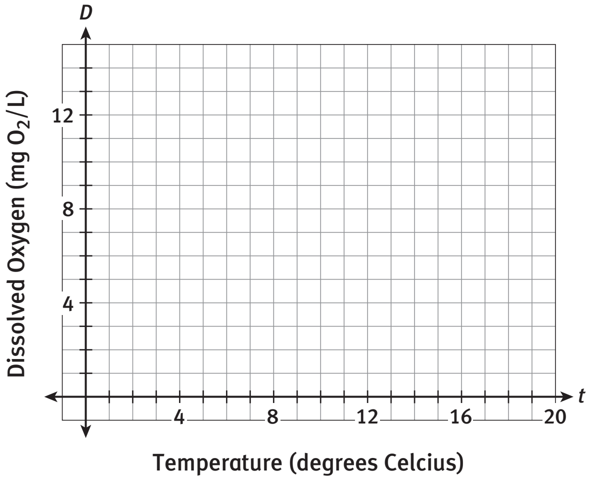
CONNECT TO ECOLOGY

Fish and other aquatic organisms need oxygen to live, just as mammals do. Dissolved oxygen in the water passes through fish gills and then is transferred into the bloodstream. When dissolved oxygen levels in the water are too low, then not enough oxygen will move into the bloodstream of the fish to maintain life.

The amount of dissolved oxygen in a body of water decreases as the water temperature increases. Dissolved oxygen needs to be at sufficient levels to sustain the life of aquatic organisms such as fish. The table shows the temperature *t* and the corresponding amount of dissolved oxygen *D* in a stream that flows into Lake Superior on several dates from May to August.

Date	*t*° (Celsius)	D (mg O_2/L)	
May 1	11.5	10.6	
May 15	12.5	9.8	
June 1	13.0	9.5	
June 15	14.0	8.7	
July 1	14.5	8.5	
July 15	15.0	8.1	
Aug 1	16.5	7.4	

1. Graph the data above as a set of points on the axes.



2. Are these data linear? Explain why or why not.

3. Add a fourth column to the table, showing the product of *t* and *D*.

4. What do you observe about the products of *t* and *D* that you recorded in the table?

My Notes

SUGGESTED LEARNING STRATEGIES: Note Taking, Interactive Word Wall, Vocabulary Organizer, Summarize/Paraphrase/Retell, Think/Pair/Share, Create Representations, Quickwrite, Work Backward

ACADEMIC VOCABULARY

inverse variation

Inverse Variation Equation

When the product of two variable quantities x and y is constant, the two variables are said to vary inversely.

If $xy = k$ and $x \neq 0$, then $y = \frac{k}{x}$, where k is the **constant of variation**.

Although the products of t and D from Item 3 are not constant, the products are close in value. When you use mathematics to model a real-world situation, the functions do not always give exact results.

5. If you use inverse variation to model the dissolved oxygen and temperature relationship, what value would you choose for k?

6. Write an inverse variation equation relating t and D that shows a constant product. Then solve the equation for D.

7. Use your calculator to make a scatter plot of the points (t, D) and graph the equation from Item 6 on the axes in Item 1.

TECHNOLOGY TIP

Use a graphing calculator to compute a power regression for the data and compare that equation to the one you wrote.

8. How well does the model that you created fit the data?

9. When dissolved oxygen is less than 6 mg O_2/L, salmon are in danger. Use the model to find the maximum safe temperature for salmon.

SUGGESTED LEARNING STRATEGIES: Create Representations, Marking the Text, Group Presentation, Vocabulary Organizer, Interactive Word Wall, Note Taking

My Notes

TRY THESE A

Use an inverse variation equation to solve each problem.

a. y varies inversely as x. When x is 5, y is 10. Find y when x is 18.

b. The length of a rectangle varies inversely as its width. If the area is 40 in.^{2} and the width is 12.5 in., find the length of the rectangle.

c. Boyle's law says that the volume of a gas in a closed container at constant temperature is inversely proportional to the pressure of the gas. Suppose 5 L of a gas are at a pressure of 2.0 atmospheres. What will be the volume if the pressure is increased to 3.0 atmospheres?

Another type of variation is **direct variation**. Two unknowns x and y vary directly if they are related by the equation $\boldsymbol{y = kx}$ where k is a nonzero constant. The graph of a direct variation equation is a line passing through the origin.

MATH TERMS

Other types of variation include **joint variation**, written in the form $y = kxz$, and **combined variation**, written in the form $y = \frac{kx}{z}$. The constant of variation is k in both types of variation.

EXAMPLE 1

The area of a rectangle with a fixed width varies directly as its length. When the area is 40 cm^2, the length is 5 cm. Write a direct variation equation for the area of the rectangle. Use the equation to determine the area when the length is 20 cm.

Step 1: *Use the direct variation formula with A as area, l as length, k as constant of proportionality.* $A = kl$

Step 2: *Substitute A = 40 and l = 5 to find k.* $40 = 5k$, $k = 8$

Step 3: *Write the direct variation equation for this situation.* $A = 8l$

Step 4: *Find A when l = 20.* $A = 8(20)$, $A = 160$

Solution: The area is 160 cm^2.

TRY THESE B

Use a direct variation equation to solve the problem.

a. y varies directly as x. When $x = 3$, $y = 30$. Find y when $x = 7$.

b. Distance traveled varies directly as time if the speed is constant. A 500-mi trip takes 8 h at a constant speed. How long would it take to travel 400 mi at the same speed?

SUGGESTED LEARNING STRATEGIES: Vocabulary Organizer, Interactive Word Wall, Create Representations, Quickwrite, Group Presentation

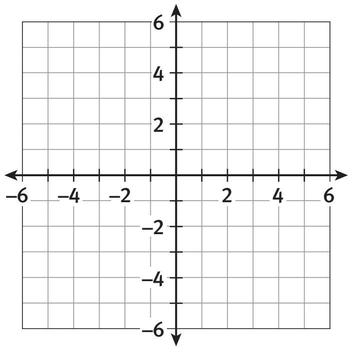
My Notes

MATH TIP

The basic rational function is sometimes called the *reciprocal function*. It can be easily graphed by plotting the ordered pairs $\left(n, \frac{1}{n}\right)$.

The rational function $f(x) = \frac{1}{x}$ is an example of an inverse variation equation whose constant of variation is 1.

10. Make a table of values in the My Notes section. Graph the parent rational function $f(x) = \frac{1}{x}$ below.



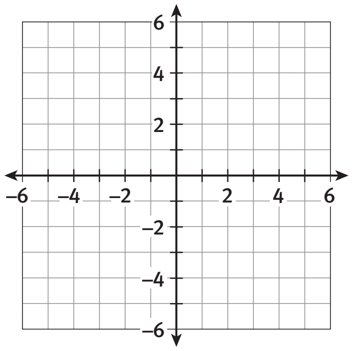
11. Describe the key features of $f(x) = \frac{1}{x}$. Use appropriate mathematics vocabulary in your description.

Functions like the one modeling dissolved oxygen and temperature are a *vertical stretch* of the parent graph $f(x) = \frac{1}{x}$.

MATH TIP

Given $y = f(x)$, the function $y = c \cdot f(x)$ represents a *vertical stretch or shrink* of the original function whose y-values have been multiplied by $c > 0$. You have a vertical stretch for $c > 1$ and a vertical shrink for $0 < c < 1$.

12. Enter the functions $f(x) = \frac{1}{x}$, $g(x) = \frac{2}{x}$, and $h(x) = \frac{5}{x}$ into your graphing calculator. Sketch the graphs on the axes below.



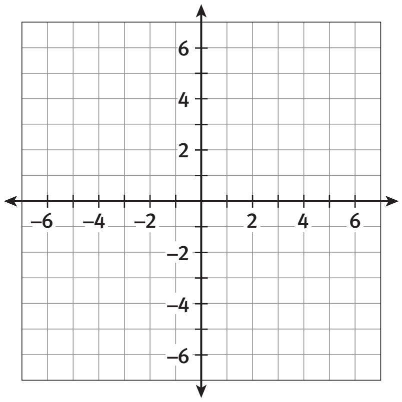
SUGGESTED LEARNING STRATEGIES: Look For a Pattern, Quickwrite, Create Representations, Predict and Confirm, Vocabulary Organizer, Interactive Word Wall

My Notes

13. How do the y-values of g and h compare to those of the parent graph?

14. Describe the similarities and the differences in the graphs of those three functions.

15. Sketch the parent graph $f(x) = \frac{1}{x}$ and the graph of $k(x) = \frac{3}{x}$ on the same axes without using your graphing calculator.



16. Without using your calculator, predict what the graph of $f(x) = -\frac{1}{x}$ will look like. Confirm prediction by graphing both functions on your calculator.

MATH TIP

Given $y = f(x)$, the function $y = -f(x)$ represents a *vertical reflection* of the original function whose y-values have been multiplied by -1.

SUGGESTED LEARNING STRATEGIES: Create Representations, Look For a Pattern, Note Taking, Vocabulary Organizer, Interactive Word Wall

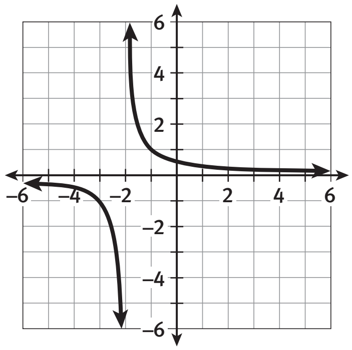
My Notes

MATH TERMS

Given $y = f(x)$, the function $y = f(x + c)$ results in a *horizontal translation* of the original function and $y = f(x) + c$ results in a *vertical translation* of the original function

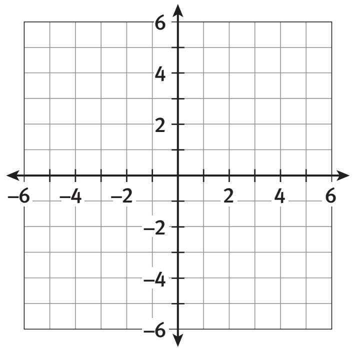
17. Sketch the graph of each function and then describe it as a transformation of the parent graph $f(x) = \frac{1}{x}$. The first graph has been done for you.

$f(x) = \frac{1}{x+2}$



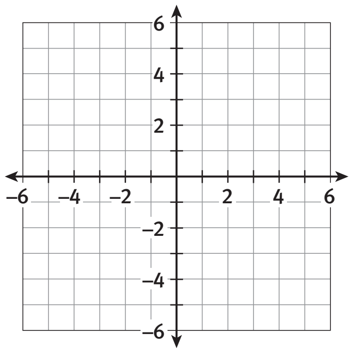
Transformation:

$f(x) = \frac{1}{x-2}$



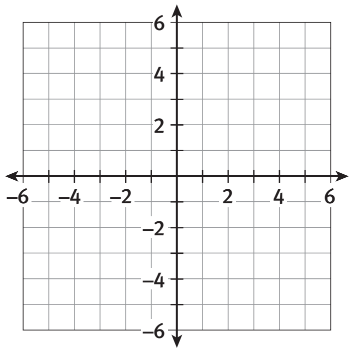
Transformation:

$f(x) = \frac{1}{x} + 2$



Transformation:

$f(x) = \frac{1}{x} - 2$



Transformation:

SUGGESTED LEARNING STRATEGIES: Create Representations, Look for a Pattern, Quickwrite, Note Taking, Identify a Subtask, Graphic Organizer

My Notes

TRY THESE C

Describe each function as a transformation of $f(x) = \frac{1}{x}$.

a. $f(x) = \frac{1}{x+1}$

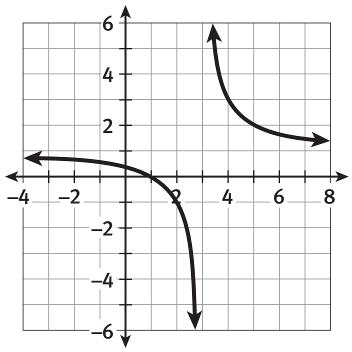
b. $f(x) = \frac{1}{x} - 3$

c. $f(x) = \frac{1}{x-5} + 3$

EXAMPLE 2

Describe the function $f(x) = \frac{2}{x-3} + 1$ as a transformation of $f(x) = \frac{1}{x}$.

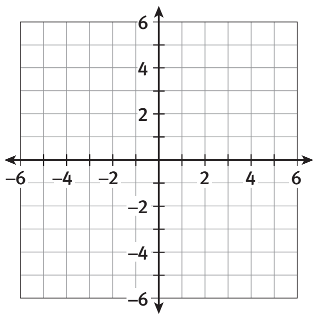
Identify the x- and y-intercepts and the asymptotes. Sketch the graph.

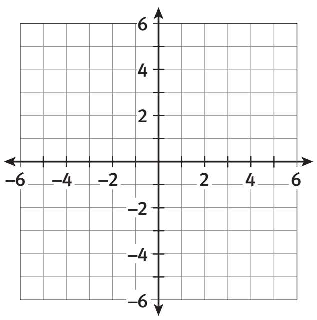
Transformations	**Asymptotes**
• vertical stretch by a factor of 2 • horizontal translation 3 units to the right • vertical translation 1 unit up	$x = 3$ $y = 1$
Intercepts y-intercept: $f(0) = \frac{1}{3}$ x-intercept: Solve $f(x) = 0$. $\frac{2}{x-3} + 1 = 0$ $\frac{2}{x-3} = -1$ $2 = -1(x-3)$ $x = 1$	**Graph** 

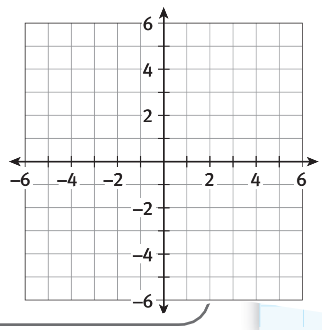
TRY THESE D

On a separate sheet of grid paper, describe each function as a transformation of $f(x) = \frac{1}{x}$. Identify the x- and y-intercepts and the asymptotes. Sketch the graph.

a. $f(x) = \frac{3}{x} + 1$ **b.** $f(x) = -\frac{1}{x+1} - 2$ **c.** $f(x) = 3 + \frac{4}{x-2}$







CHECK YOUR UNDERSTANDING

Write your answers on notebook paper or grid paper. Show your work.

1. y varies inversely as x. When $y = 5$, $x = 20$. Find y when $x = 5$.

In Items 2–4, the time to travel a fixed distance varies inversely as speed.

2. Write an inverse variation model for a trip that takes 1.5 h when you average 50 mph.
3. Interpret the meaning of the constant of variation.
4. Use the equation to determine how long the trip takes if you average 65 mph.
5. Does the data below represent an inverse variation model? Explain why or why not.

x	2	4	6	8
y	24	12	8	6

6. Does the data below represent a direct variation model? Explain why or why not.

x	2	4	6	8
y	6	10	14	18

7. y varies directly as x. When $y = 5$, $x = 20$. Find y when $x = 5$.

Describe each function as a transformation of $f(x) = \frac{1}{x}$.

8. $f(x) = \dfrac{2}{x+5}$
9. $f(x) = -\dfrac{1}{x-3} + 5$
10. $f(x) = \dfrac{10}{x+4} - 2$

Graph each rational function, using your knowledge of transformations.

11. $f(x) = \dfrac{1}{x-3} + 1$
12. $f(x) = -\dfrac{2}{x} - 4$
13. MATHEMATICAL REFLECTION How can you use transformations to graph rational functions?

Simplifying Rational Expressions
It's All Rational

ACTIVITY
5.6

SUGGESTED LEARNING STRATEGIES: Summarize/Paraphrase/Retell, Interactive Word Wall, Vocabulary Organizer, Note Taking, Think/Pair/Share, Create Representations, Simplify the Problem

My Notes

MATH TIP

When a rational function has a denominator of zero, restrictions on the variable are needed.

Rational expressions can be simplified and combined, using the operations of addition, subtraction, multiplication and division.

Writing rational expressions in simpler forms and combining them helps you to understand and graph rational functions and solve equations.

To simplify a rational expression, factor the numerator and denominator. Identify the restrictions on the variable x that make the denominator in the expression equal to zero. Then, divide out the common factors.

EXAMPLE 1

Simplify each expression.

		A.	B.
	Original expression	$\dfrac{x^2 + 5x - 14}{x^2 - 4}$	$\dfrac{2x^2 + 7x + 3}{x^2 + 7x + 12}$
Step 1:	*Identify the restrictions on* x. *Set the denominators equal to zero.*	$x^2 - 4 = 0$ $(x + 2)(x - 2) = 0$ $x + 2 = 0$ or $x - 2 = 0$ $x = -2$ or $x = 2$	$x^2 + 7x + 12 = 0$ $(x + 3)(x + 4) = 0$ $x + 3 = 0$ or $x + 4 = 0$ $x = -3$ or $x = -4$
Step 2:	*Factor the numerators and denominators.*	$\dfrac{x^2 + 5x - 14}{x^2 - 4} =$ $\dfrac{(x + 7)(x - 2)}{(x + 2)(x - 2)} =$	$\dfrac{2x^2 + 7x + 3}{x^2 + 7x + 12} =$ $\dfrac{(2x + 1)(x + 3)}{(x + 4)(x + 3)} =$
Step 3:	*Divide out common factors.*	$\dfrac{(x + 7)\cancel{(x - 2)}}{(x + 2)\cancel{(x - 2)}} =$ $\dfrac{x + 7}{x + 2}, x \neq 2, -2$	$\dfrac{(2x + 1)\cancel{(x + 3)}}{(x + 4)\cancel{(x + 3)}} =$ $\dfrac{2x + 1}{x + 4}, x \neq -3, -4$

TRY THESE A

Simplify. Identify any restrictions on x. Write your answers on notebook paper. Show your work.

a. $\dfrac{x^2 + 20x + 36}{x^3 - 4x}$

b. $\dfrac{x^2 - 2x - 15}{2x^2 + 3x - 9}$

c. $\dfrac{x^3 - 9x}{3 - x}$

MATH TIP

Make sure that you use the original rational expression when identifying restrictions on the variable.

SUGGESTED LEARNING STRATEGIES: Summarize/Paraphrase/Retell, Note Taking, Marking the Text, Think/Pair/Share, Create Representations, Group Presentation

My Notes

To multiply rational expressions and express the product in lowest terms, factor the numerator and denominator of each expression. Then, divide out any common factors.

EXAMPLE 2

Multiply the expression. Assume no denominator is zero.

Original expression $\frac{2x^2-8}{x^2-1} \cdot \frac{x^2+2x+1}{x^3-x^2-2x}$

Step 1: *Factor the numerators and denominators.* $\frac{2(x+2)(x-2)}{(x+1)(x-1)} \cdot \frac{(x+1)(x+1)}{x(x-2)(x+1)}$

Step 2: *Divide out common factors.* $\frac{2(x+2)\cancel{(x-2)}\cancel{(x+1)}\cancel{(x+1)}}{\cancel{(x+1)}(x-1)x\cancel{(x-2)}\cancel{(x+1)}}$

$\frac{2(x+2)}{x(x-1)}$

MATH TIP

When dividing numerical fractions, write as multiplication.

$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$

If a, b, c, and d have any common factors, you can cancel them before you multiply.

$\frac{4}{15} \div \frac{8}{3} = \frac{4}{15} \cdot \frac{3}{8} =$

$\frac{\cancel{4}}{\cancel{3} \cdot 5} \cdot \frac{\cancel{3}}{2 \cdot \cancel{4}} = \frac{1}{10}$

To divide rational expressions, write division as multiplication and then finish simplifying the expression.

EXAMPLE 3

Divide the expression. Assume no denominator is zero.

Original expression $\frac{x^2+5x+6}{x^2-4} \div \frac{5x+15}{3x^2-4x-4}$

Step 1: *Write as multiplication.* $\frac{x^2+5x+6}{x^2-4} \cdot \frac{3x^2-4x-4}{5x+15}$

Step 2: *Factor the numerators and denominators.* $\frac{(x+2)(x+3)}{(x+2)(x-2)} \cdot \frac{(3x+2)(x-2)}{5(x+3)}$

Step 3: *Divide out common factors.* $\frac{\cancel{(x+2)}\cancel{(x+3)}(3x+2)\cancel{(x-2)}}{\cancel{(x+2)}\cancel{(x-2)}(5)\cancel{(x+3)}}$

$\frac{3x+2}{5}$

TRY THESE B

Perform the indicated operation. Assume no denominator is zero. Write your answers on notebook paper. Show your work.

a. $\frac{2x+4}{x^2-25} \cdot \frac{x^2-5x-50}{4x^2-16}$

b. $\frac{6x^2}{3x^2-27} \div \frac{2x+2}{x^2-2x-3}$

SUGGESTED LEARNING STRATEGIES: Summarize/Paraphrase/Retell, Note Taking, Graphic Organizer, Think/Pair/Share

My Notes

To add or subtract rational expressions with unlike denominators, find a common denominator. The easiest way to find the least common denominator is to factor the expressions. Then, the least common denominator is the product of each factor common to the expressions and any non-common factors.

EXAMPLE 4

Find the least common denominator of $\frac{1}{x^2 - 3x - 4}$ and $\frac{1}{x^2 - 16}$.

Step 1: *Factor each denominator.* $x^2 - 3x - 4 = (x + 1)(x - 4)$
$x^2 - 16 = (x + 4)(x - 4)$

Step 2: *Identify common factors and factors not in common.* Factors in Common: $x - 4$

Factors Not in Common: $x + 4, x + 1$

Step 3: *Write the least common denominator.* $(x + 4)(x + 1)(x - 4)$

TRY THESE C

Find the least common denominator of $\frac{1}{x^2 - 9}$ and $\frac{1}{3x^2 - 9x}$.

Now you are ready to add and subtract rational expressions with different denominators.

EXAMPLE 5

Simplify the expression. Assume no denominator is zero.

Original expression $\frac{2}{x-2} - \frac{3}{x^2 - 2x} =$

Step 1: *Factor the denominators.* $\frac{2}{x-2} - \frac{3}{x(x-2)}$

Step 2: *Find the least common denominator.* $x(x - 2)$

Step 3: *Multiply numerator and denominator of each term by the missing factor(s) of the least common denominator.* $\frac{2(x)}{x(x-2)} - \frac{3}{x(x-2)} =$

Step 4: *Subtract the like fractions to find the solution.* $\frac{2x - 3}{x(x-2)}$

MATH TIP

When the denominators are the same, all you have to do is add or subtract the numerators as indicated by the operation.

CONNECT TO AP

You will continue to use the skill of simplifying rational expressions in AP Calculus.

My Notes

SUGGESTED LEARNING STRATEGIES: Think/Pair/Share, Simplify the Problem, Group Presentation, Summarize/Paraphrase/Retell, Vocabulary Organizer, Interactive Word Wall, Note Taking, Create Representations

TRY THESE D

Simplify each expression. Assume no denominator is zero. Write your answers on notebook paper. Show your work.

a. $\dfrac{3}{x+1} - \dfrac{x}{x-1}$ **b.** $\dfrac{2}{x} - \dfrac{3}{x^2-3x}$ **c.** $\dfrac{2}{x^2-4} + \dfrac{x}{x^2+4x+4}$

ACADEMIC VOCABULARY

A rational expression that contains rational expressions in its numerator and/or its denominator is called a **complex fraction**.

You can simplify **complex fractions** if you treat them like a division problem. Simplify the numerator and denominator as much as possible, and then write the problem using multiplication.

EXAMPLE 6

Simplify $\dfrac{1+\dfrac{1}{x+1}}{x-\dfrac{x}{x-1}}$. Assume no denominator is zero.

Original expression $\qquad \dfrac{1+\dfrac{1}{x+1}}{x-\dfrac{x}{x-1}} =$

Step 1: *Simplify the numerator and denominator using their least common denominators.* $\qquad \dfrac{\dfrac{x+1}{x+1}+\dfrac{1}{x+1}}{\dfrac{x(x-1)}{(x-1)}-\dfrac{x}{x-1}} =$

Step 2: *Add or subtract fractions in the numerator and denominator. Combine like terms.* $\qquad \dfrac{\dfrac{x+1+1}{x+1}}{\dfrac{x^2-x-x}{x-1}} = \dfrac{\dfrac{x+2}{x+1}}{\dfrac{x^2-2x}{x-1}} =$

Step 3: *Write division as multiplication of the reciprocal.* $\qquad \dfrac{x+2}{x+1} \cdot \dfrac{x-1}{x^2-2x} =$

Step 4: *Factor and simplify if possible.* $\qquad \dfrac{(x+2)(x-1)}{x(x+1)(x-2)}$

TRY THESE E

Simplify. Assume no denominator is zero. Write your answers on notebook paper. Show your work.

a. $\dfrac{\dfrac{x^2-3x-4}{x^2-4}}{\dfrac{2x^2+2x}{x+2}}$ **b.** $\dfrac{\dfrac{x}{x+1}-\dfrac{1}{x-1}}{\dfrac{1}{x+1}+2}$

SUGGESTED LEARNING STRATEGIES: Vocabulary Organizer, Note Taking, Think/Pair/Share, Look for a Pattern, Marking the Text, Shared Reading

My Notes

In the graph of a rational function, a break in the graph often signals that a **discontinuity** has occurred. Algebraically, a discontinuity happens for values of x that cause the function to be undefined and are therefore not in the domain of the function.

EXAMPLE 7

Identify any vertical asymptotes in the graph.

Step 1: *Factor the numerator and denominator.*

$$f(x) = \frac{x^2 - 4}{x^2 + 5x + 6}$$

$$f(x) = \frac{(x + 2)(x - 2)}{(x + 2)(x + 3)}$$

Step 2: *Divide out the common factors.*

$$f(x) = \frac{x - 2}{x + 3}$$

Step 3: *Find the values that make the simplified denominator = 0.*

$x + 3 = 0$ when $x = -3$

vertical asymptote $x = -3$

TRY THESE F

Identify any vertical asymptotes in the graph.

a. $f(x) = \dfrac{x^2 - x}{x^2 + 3x - 4}$

b. $f(x) = \dfrac{3 - x}{9 - x^2}$

MATH TIP

To find the vertical asymptote of a graph, determine the values of the variables that make the function undefined when it is in simplest form.

A *horizontal asymptote* depends on the degrees of the numerator and denominator and describes the end behavior of a rational function.

- When the degrees are the same, the horizontal asymptote is the ratio of the leading coefficients.
- When the denominator degree is larger, the horizontal asymptote is equal to 0.
- When the numerator degree is larger, there is no horizontal asymptote.

TECHNOLOGY TIP

Use a graphing calculator to graph the function and visually see the breaks where the asymptotes are located.

EXAMPLE 8

Identify the horizontal asymptote, if any.

a. $f(x) = \dfrac{2 + x}{x^2 - 1}$

numerator degree $= 1$
denominator degree $= 2$

$2 > 1$

horizontal asymptote: $y = 0$

b. $f(x) = \dfrac{2x + 2}{x - 1}$

numerator degree $= 1$
denominator degree $= 1$

lead coefficients: 2, 1
ratio of lead coefficients: 2

horizontal asymptote: $y = 2$

SUGGESTED LEARNING STRATEGIES: Think/Pair/Share, Look for a Pattern, Group Presentation, Shared Reading, Questioning the Text, Graphic Organizer, Note Taking

My Notes

TRY THESE G

Identify the horizontal asymptote, if any.

a. $f(x) = \dfrac{2-x}{x+4}$ **b.** $f(x) = \dfrac{x^2-1}{x+3}$ **c.** $\dfrac{x}{x^2-4}$

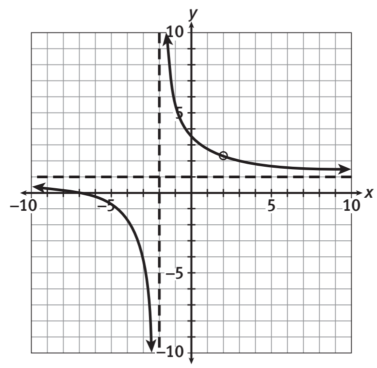
Now you are ready to use your knowledge of simplifying rational expressions to help you understand and graph rational functions.

To graph rational functions, follow these steps.

- Simplify the rational function.
- Express the numerator and denominator in factored form.
- Identify vertical asymptotes.
- Identify x- and y-intercepts.
- Identify horizontal asymptote (end behavior).
- Make a sketch, using a graphing calculator as needed.

EXAMPLE 9

Analyze and graph the rational function $f(x) = \dfrac{x^2+5x-14}{x^2-4}$.

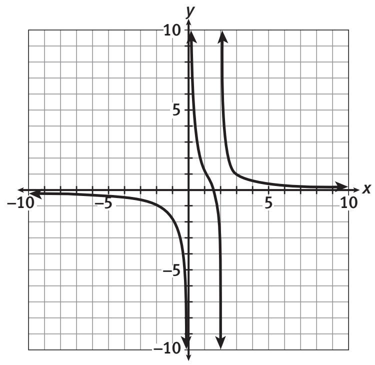
Simplify. $\dfrac{x^2+5x-14}{x^2-4} =$ $\dfrac{(x+7)(x-2)}{(x+2)(x-2)} =$ $\dfrac{(x+7)\cancel{(x-2)}}{(x+2)\cancel{(x-2)}} =$ $\dfrac{x+7}{x+2}$	**Identify vertical asymptotes.** $x+2=0$, so vertical asymptote is $x=-2$ **Identify intercepts.** x-intercept: $x+7=0$, so $x=-7$ y-intercept: $f(0) = \dfrac{0+7}{0+2} = 3.5$
Identify horizontal asymptote. numerator degree = 1 denominator degree = 1 lead coefficients: 1, 1 ratio of lead coefficients: 1 horizontal asymptote: $y=1$	**Graph.** 

SUGGESTED LEARNING STRATEGIES: Graphic Organizer, Note Taking, Identify a Subtask, Create Representations, Group Presentation

My Notes

EXAMPLE 10

Analyze and graph the rational function $f(x) = \frac{2}{x-2} - \frac{3}{x^2 - 2x}$.

Simplify. $\frac{2}{x-2} - \frac{3}{x^2-2x} =$ $\frac{2}{x-2} - \frac{3}{x(x-2)} =$ $\frac{2(x)}{x(x-2)} - \frac{3}{x(x-2)} =$ $\frac{2x-3}{x(x-2)}$	**Identify vertical asymptotes.** $x = 0$ and $x - 2 = 0$, so vertical asymptotes are $x = 0$ and $x = 2$ **Identify intercepts.** x-intercept: $2x - 3 = 0$, so $x = 1.5$ y-intercept: none, because $f(x)$ is undefined when $x = 0$
Identify horizontal asymptote. numerator degree $= 1$ denominator degree $= 2$ horizontal asymptote is $y = 0$	**Graph.** 

TRY THESE **H**

Analyze and graph each rational function. Write your answers on grid paper. Show your work.

a. $f(x) = \frac{x^2 - 4}{x^3 - 3x^2 - 10x}$

b. $f(x) = \frac{1}{x+1} - \frac{2}{x+3}$

MATH TIP

To determine when a sum or difference of rational expressions is 0 or undefined, it helps to combine them into a single expression first.

My Notes

CHECK YOUR UNDERSTANDING

Write your answers on notebook paper or grid paper. Show your work.

Simplify. Identify any restrictions on x.

1. $\dfrac{2x^2 + 5x + 2}{x^2 - 4}$

2. $\dfrac{16 - x^2}{x^3 + 6x^2 + 8x}$

Perform the indicated operation. Assume no denominator is equal to zero.

3. $\dfrac{x^2 - 5x - 6}{x^2 - 12x + 36} \cdot \dfrac{x^2 - 36}{x^2 - 19x - 20}$

4. $\dfrac{2x^2 + 3x + 1}{x^2 - 1} \cdot \dfrac{2x + 1}{4x^2 + 4x + 1}$

5. $\dfrac{3x^2 + 4x - 4}{2x - 4} \div \dfrac{9x^2 - 4}{3x^2 - 7x - 6}$

Find the simplest common denominator.

6. $\dfrac{1}{5x + 10}$ and $\dfrac{2}{x^2 + 4x + 4}$

7. $\dfrac{1}{x - 3}$, $\dfrac{x}{x^2 - 6x + 9}$, and $\dfrac{2x}{x^2 + 7x - 30}$

Perform the indicated operation. Assume no denominator is equal to zero.

8. $\dfrac{2}{x + 3} - \dfrac{x}{x - 1}$

9. $\dfrac{2}{x^2 - 3x - 4} - \dfrac{1}{x^2 - 1}$

10. For each function, identify any vertical asymptotes and horizontal asymptotes.

a. $f(x) = \dfrac{x^2 - 25}{x^2 - 2x - 35}$

b. $f(x) = \dfrac{2x + 4}{x^2 - 4}$

Analyze and graph each rational function.

11. $f(x) = \dfrac{x^2 + 2x}{x^2 - x - 6}$

12. $f(x) = \dfrac{x}{x + 1} + \dfrac{1}{x - 1}$

13. **MATHEMATICAL REFLECTION** What have you learned about simplifying rational expressions and graphing rational functions as a result of this activity?

Rational Equations and Inequalities
A Rational Pastime

ACTIVITY **5.7**

SUGGESTED LEARNING STRATEGIES: Marking the Text, Summarize/Paraphrase/Retell, Questioning the Text, Create Representations, Note Taking, Identify a Subtask, Think/Pair/Share

My Notes

Jesse pitches for the baseball team and wants to improve his batting average before the county all-stars are selected. To date, he has 10 hits out of 40 times at the bat.

1. Batting average is the ratio of hits to at-bats. Write a ratio that represents Jesse's current batting average for this season and express the ratio in decimal form.

CONNECT TO MEASUREMENT

When a ratio is formed by two quantities with different units, it is also called a *rate*. Batting average is a rate, and even though we call it an average, it does not represent the mean of a set of numbers.

Jesse wants to improve his batting average to at least 0.320. If he gets a hit every time he bats, then his new batting average would be $\frac{10 + x}{40 + x}$ where x is the number of future hits in as many times at-bat.

2. Write an equation to determine how many consecutive hits he needs to bat 0.320.

To solve equations like the one you wrote in Item 2, multiply by an expression that eliminates all the denominators.

EXAMPLE 1

Solve $\frac{x^2 - 4}{x + 1} = x + 5$.

Original equation, undefined at $x = -1$

$$\frac{x^2 - 4}{x + 1} = x + 5$$

Step 1: *Multiply both sides by $(x + 1)$ to cancel the denominator.*

$$(x + 1)\left(\frac{x^2 - 4}{x + 1}\right) = (x + 5)(x + 1)$$

Step 2: *Solve for x.*

$$x^2 - 4 = x^2 + 6x + 5$$
$$-4 = 6x + 5$$
$$6x = -9$$
$$x = -1.5$$

Step 3: *Check to see if the original equation is undefined at the solution.*

3. Solve the equation you wrote in Item 2 to find the number of consecutive hits that Jesse needs to increase his batting average.

My Notes

SUGGESTED LEARNING STRATEGIES: Note Taking, Think/Pair/Share, Group Presentation, Vocabulary Organizer, Interactive Word Wall, Identify a Subtask, Guess and Check

EXAMPLE 2

Solve $\frac{2}{x} - \frac{1}{x+2} = \frac{3}{x}$.

Original equation, undefined at $x = 0$ and $x = -2$

$$\frac{2}{x} - \frac{1}{x+2} = \frac{3}{x}$$

Step 1: *Multiply both sides by $x(x+2)$ to cancel the denominators.*

$$x(x+2)\left(\frac{2}{x} - \frac{1}{x+2}\right) = \left(\frac{3}{x}\right)x(x+2)$$

Step 2: *Solve for x.*

$$\begin{aligned} 2(x+2) - 1(x) &= 3(x+2) \\ 2x + 4 - x &= 3x + 6 \\ x + 4 &= 3x + 6 \\ -2x &= 2 \\ x &= -1 \end{aligned}$$

Step 3: *Check to see if the original equation is undefined at the solution.*

TRY THESE A

Solve each equation and check your solution.

a. $\frac{x+4}{x+5} = \frac{3}{5}$

b. $\frac{2x}{x+2} - \frac{x}{x-1} = 1$

MATH TIP

When checking your solutions, substitute the solution into the original equation.

When solving a rational equation, it is possible to introduce an *extraneous solution*. An **extraneous solution** is not valid in the original equation although it satisfies the polynomial equation that results when you multiply by the simplest common denominator.

4. Solve the equation $\frac{1}{x} - \frac{2x}{x+2} = \frac{x-6}{x(x+2)}$.

5. Identify any extraneous solutions to the equation in Item 4.

TRY THESE B

Solve each equation. Identify any extraneous solutions. Write your answers on notebook paper. Show your work.

a. $\frac{x}{x-1} = \frac{1}{x-1} + \frac{2}{x}$

b. $\frac{1}{x} - \frac{x-1}{x^2+x} = \frac{x-1}{x+1}$

SUGGESTED LEARNING STRATEGIES: Shared Reading, Questioning the Text, Think/Pair/Share, Simplify the Problem, Create Representations, Graphic Organizer, Identify a Subtask

My Notes

Jesse's coach requires the team to help prepare the baseball diamond at school. Jesse and Cody working together can clean up the infield in 2 h. If Jesse worked alone, it would take him 5 h. To figure out how long it would take Cody to prepare the infield by himself, you must consider the portion of the job that can be completed in 1 h.

6. If Jesse takes 5 hours to complete the job, then what fraction could he complete in 1 hour, assuming he works at an even pace?

7. If it takes Cody *t* hours to complete the job, then what fraction could he complete in 1 hour assuming he works at an even pace?

8. Write a similar fraction for the amount of work done in 1 hour when both boys work together.

9. Now write an equation using the verbal model below.

Jesse's work in 1 hour	+	Cody's work in 1 hour	=	Together work in 1 hour

10. Solve the equation you wrote in Item 9 to determine how long it would take Cody to complete the job if he worked alone.

11. Garrett has cleaned up the field on his own and it took him 4 hours. How long will it take all three boys working together to prepare the infield for a game?

SUGGESTED LEARNING STRATEGIES: Questioning the Text, Identify a Subtask, Look for a Pattern, Quickwrite

My Notes

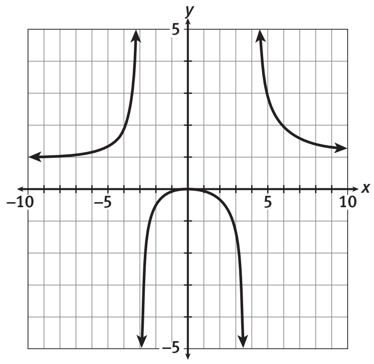
The rational inequality shown below can be solved graphically or numerically.

$$\frac{x^2 - 1}{x^2 - x - 12} < 0$$

12. First, factor the left side of the inequality and determine the zeros and the values of x that are not in the domain of the function.

13. The graph of the left side of the inequality is shown below. The table shows the x- and y-coordinates and the sign of y.

x	y	sign
−5	1.333	+
−4	1.875	+
−3	undefined	
−2	−0.5	−
−1	0	
0	0.083	+
1	0	
2	−0.3	−
3	−1.333	−
4	undefined	
5	3	+



a. Identify the intervals of x where the graph is below the x-axis.

b. Look back to the original inequality. Why would the intervals of x where the graph is below the x-axis be the solutions to the inequality?

SUGGESTED LEARNING STRATEGIES: Notetaking, Graphic Organizer, Identify a Subtask, Look for a Pattern, Create Representations

My Notes

You can solve rational inequalities without using tables and graphs.

Solving Rational Inequalities

- Write the inequality in factored form.
- Identify the zeros of the numerator and the zeros of the denominator. (Note that the zeros of the denominator are the values where the rational function is not defined.)
- Pick one test value for x that falls between each of the zeros.
- Evaluate the left hand side of the inequality at these values to test the sign of the inequality in each interval and determine the solution.
- State the solution intervals and graph them on the number line.

MATH TIP

To use this method, the inequality must be set up so that one side is a single rational expression and the other side is 0. If it is not, first transform the inequality into that form.

EXAMPLE 3

Solve the inequality $\frac{x^2 - 1}{x^2 - 2x - 8} \le 0$.

Factor: $\frac{(x + 1)(x - 1)}{(x - 4)(x + 2)} \le 0$

Zeros of the numerator at $x = 1$ and -1
Zeros of the denominator (where function is undefined) at $x = -2$ and 4

The zeros, in order from least to greatest are: $-2, -1, 1, 4$
Pick and test one value in each interval: $-3, -1.5, 0, 2$, and 5

For $x = -3$:

$$\frac{(-3 + 1)(-3 - 1)}{(-3 - 4)(-3 + 2)} = \frac{(-2)(-4)}{(-7)(-1)} > 0$$

For $x = -1.5$:

$$\frac{(-1.5 + 1)(-1.5 - 1)}{(-1.5 - 4)(-1.5 + 2)} = \frac{(-0.5)(-2.5)}{(-5.5)(0.5)} < 0$$

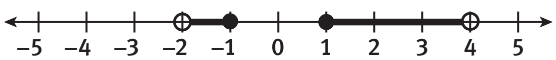
Continue this process and record the results in a table.

interval	$x < -2$	$-2 < x \le -1$	$-1 \le x \le 1$	$1 \le x < 4$	$x > 4$
test value	-3	-0.5	0	2	5
sign	$+$	$-$	$+$	$-$	$+$

The solution is the intervals of x where the inequality was less than or equal to 0 (recall the "≤ 0" in the original inequality). So, x-values of the numerator zeros are included in the solution.

Solution intervals: $-2 < x \le -1$ or $1 \le x < 4$

Graph the solution on a number line.



MATH TIP

You do not need to evaluate the inequality completely. Simply plug in the test values and figure out the sign of each factor and then the overall sign of the rational expression.

My Notes

SUGGESTED LEARNING STRATEGIES: Graphic Organizer, Identify a Subtask, Look for a Pattern, Guess and Check, Create Representations, Group Presentation

TRY THESE C

Solve each inequality.

a. $\dfrac{x^2 - 5x - 6}{x^2 - 4x + 3} \geq 0$

b. $\dfrac{1}{x} - \dfrac{2}{x + 2} < 0$

CHECK YOUR UNDERSTANDING

Write your answers on notebook paper or grid paper. Show your work.

Solve each equation. Identify any extraneous solutions.

1. $\dfrac{x + 3}{2} = \dfrac{5}{x}$

2. $\dfrac{2}{x} + \dfrac{3}{x} = \dfrac{5}{x + 1}$

3. $\dfrac{x - 3}{x - 1} - \dfrac{2}{x + 1} = \dfrac{x - 5}{x^2 - 1}$

4. $\dfrac{1}{x - 3} = \dfrac{x}{9 - 3x}$

5. A chemist has 100 units of a 10% solution and wants to strengthen it to 30%. How much pure chemical should be added to the original solution to achieve the desired concentration?

6. Raj, Ebony, and Jed paint houses during the summer. Raj takes 5 hours to paint a room by himself while it takes Ebony 4 hours and Jed 3 hours. How long will it take them if they work together?

Solve each inequality graphically or numerically.

7. $\dfrac{x^2 - 9}{x^2 - 4x - 5} < 0$

8. $\dfrac{1}{x + 1} \geq \dfrac{2}{x - 1}$

9. MATHEMATICAL REFLECTION What have you learned about solving rational equations and inequalities as a result of this activity?

Rational Equations and Functions

PLANNING A PROM

Embedded Assessment 2

Use after Activity 5.7.

1. Sketch the graph of the rational function $f(x) = \frac{x-1}{x+1}$. Identify the key features, such as asymptotes and intercepts. Then describe the graph as a transformation of the parent function $f(x) = \frac{1}{x}$.

2. Simplify the rational expression $\frac{x-2}{x+2} - \frac{x+1}{x-1}$. Assume no denominator is zero.

3. Solve the equation $\frac{2}{x} - \frac{2}{x+2} = 8$ in two ways and discuss the advantages and disadvantages of the methods you selected.

4. The prom committee is planning this year's prom. The costs are stated in the table below.

Music	\$800
Decorations	\$500
Ballroom	\$900
Catered Dinner	\$25/person

The committee must set aside 10 free prom tickets for a drawing the principal wants to have for students enrolled in Advanced Placement® classes. The ballroom can only hold 300 students, and the prom committee has not decided whether they want to have dinner catered at the prom. Based on past experience, the committee knows students are not willing to pay more than \$20 per ticket if food will not be provided and \$35 per ticket if food is provided.

Write a proposal for setting the prom ticket price. Be sure to include these items:

- the proposed ticket price
- the number of tickets that must be sold to break even
- the amount of money that will be made
- mathematics to support your reasoning

Embedded Assessment 2

Use after Activity 5.7.

Rational Equations and Functions

PLANNING A PROM

	Exemplary	Proficient	Emerging
Math Knowledge #1, 2, 3, 4	The student: • Identifies four key features of the graph. (1) • Simplifies the rational expression correctly. (2) • Solves the equation correctly in two ways. (3) • Uses the correct mathematics to support his/her reasoning. (4)	The student: • Identifies only three key features of the graph. • Uses the correct method to simplify the expression, but makes a computational error. • Solves the equation correctly in only one way. • The mathematical support contains computational errors.	The student: • Identifies at least one key feature of the graph. • Does not simplify the expression correctly. • Does not solve the equation correctly. • Does not use the correct mathematics to support his/her reasoning.
Problem Solving #4	The student correctly determines the number of tickets that must be sold to break even and the amount of money that will be made for the proposed ticket price. (4)	The student correctly determines the number of tickets that must be sold to break even or the amount of money that will be made for the proposed ticket price.	The student correctly determines neither the number of tickets that must be sold nor the amount of money that will be made.
Representations #1	The student sketches a correct graph. (1)	The student sketches a partially correct graph.	The student sketches an incorrect graph.
Communication #1, 3, 4	The student: • Gives a complete description of the transformation. (1) • Gives a complete discussion of the advantages and disadvantages of the methods selected. (3) • Writes a proposal that includes the four indicated items. (4)	The student: • Gives an incomplete description of the transformation. • Gives an incomplete discussion of the advantages and disadvantages of the methods selected. • Writes a proposal that includes only three indicated items.	The student: • Gives an inaccurate description of the transformation. • Does not discuss the advantages and disadvantages of the methods selected. • Writes a proposal that includes at least one of the indicated items.

ACTIVITY 5.1

1. What is the inverse of $f(x) = 2x + 7$?

a. $f^{-1}(x) = \frac{x}{2} + 7$ **b.** $f^{-1}(x) = \frac{x - 7}{2}$

c. $f^{-1}(x) = 7x + 2$ **d.** $f^{-1}(x) = -2x - 7$

2. The formula $f(x) = \frac{9}{5}x + 32$ converts x degrees Celsius to $f(x)$ degrees Fahrenheit.

a. Find the inverse of $f(x) = \frac{9}{5}x + 32$.

b. Identify 3 ordered pairs (a, b) that satisfy the equation $f(x) = \frac{9}{5}x + 32$. Show that the ordered pairs (b, a) satisfy the equation you found in Part (a).

c. What is the meaning of the ordered pairs (a, b) from part (b) in terms of temperatures? What is the meaning of the ordered pairs (b, a) from part (b) in terms of temperatures?

d. Is there a temperature that has the same numerical value in degrees Fahrenheit and degrees Celsius? Explain your reasoning.

3. What is the inverse of $f(x) = \sqrt{x - 3}$? State the domain and range of the function and its inverse.

4. Which functions are one-to-one?

a. $f(x) = x^2$ **b.** $f(x) = \log_2(x)$

c. $f(x) = x^3 - x$

5. The formula to convert from x in the Kelvin scale to $c(x)$ degrees Celsius is $c(x) = x - 273.15$.

a. Use your knowledge of composition of functions to write a formula to convert Kelvins to degrees Fahrenheit.

b. Use your knowledge of inverse functions to write a formula to convert degrees Fahrenheit to Kelvins.

c. Absolute zero is 0 Kelvin. Convert this temperature to degrees Fahrenheit.

d. Water boils at 212° Fahrenheit. Convert this temperature to Kelvins.

ACTIVITY 5.2

Describe each function as a transformation of $f(x) = \sqrt{x}$. State the domain and range.

6. $f(x) = -4\sqrt{x} + 2$ **7.** $f(x) = -3 + \sqrt{x + 2}$

Graph each function using your knowledge of transformations.

8. $f(x) = \sqrt{x + 4} - 5$ **9.** $f(x) = -3 - 2\sqrt{x - 1}$

Solve for x.

10. $\sqrt{x + 5} = 7$ **11.** $\sqrt{x + 3} = 2x - 9$

12. The radius of a circle is given by the formula $r = \sqrt{\frac{A}{\pi}}$.

a. What is r when $A = 100\pi$?

b. What is r when $A = 100$?

c. What is A when $r = 7$?

ACTIVITY 5.3

Simplify each expression.

13. $81^{-\frac{1}{4}}$ **14.** $(8x^4)^{\frac{1}{3}}$

15. $6^{\frac{2}{3}} \cdot 36^{\frac{2}{3}}$

Solve each equation. Assume $x > 0$.

16. $3x^{\frac{1}{4}} = 15$ **17.** $3x^3 + 81 = 0$

Simplify each expression.

18. $(\sqrt{x} - \sqrt{3})(\sqrt{x} + \sqrt{3})$ **19.** $\frac{25}{\sqrt[4]{5}}$

20. Use what you know about the graphs of power functions to solve this equation: $x^{\frac{1}{3}} = x^{\frac{1}{2}}$

21. Solve graphically or analytically: $2x^{\frac{2}{3}} > 6$

ACTIVITY 5.4

KitKat Kondos makes kitty condos. They have $10,000 in fixed operating costs and each kitty condo costs $12 to make.

22. Write a function that represents the cost of making x kitty condos.

23. Write a rational function that represents the cost per condo of x kitty condos.

24. Graph the cost per condo function.

25. What is the cost per condo for 500 kitty condos?

26. If the cost per condo was $13, how many condos did the company make?

27. Which function is an example of a rational function?

a. $f(x) = \frac{x}{3} - 3x^2$ **b.** $f(x) = \frac{x - 3x^2}{2}$

c. $f(x) = \frac{-3x^2}{x - 2}$ **d.** $f(x) = x - 3x^2$

Use $f(x) = \frac{x + 2}{2x + 1}$ to answer Items 28–29.

28. What is the vertical asymptote of f?

a. $x = -2$ **b.** $x = -0.5$

c. $x = 0.5$ **d.** $x = 2$

29. What is the horizontal asymptote of f?

a. $y = -2$ **b.** $y = -0.5$

c. $y = 0.5$ **d.** $y = 2$

30. Graph the rational function $f(x) = \frac{x - 2}{x + 5}$.

ACTIVITY 5.5

31. Given the inverse variation $y = \frac{10}{x}$, what is the constant of variation?

a. $k = -10$ **b.** $k = 1$

c. $k = 10$ **d.** $k = 100$

32. If y varies inversely as x, and $y = 8$ when $x = 40$, which equation models this situation?

a. $y = \frac{5}{x}$ **b.** $y = \frac{32}{x}$

c. $y = \frac{48}{x}$ **d.** $y = \frac{320}{x}$

33. Evan's video game scores vary inversely as the time spent playing. If he scores 1000 points after playing for 1 hour, how much will he score after playing 3.5 hours?

34. The number of pages that Emma reads varies directly as the time spent reading. If Emma reads 120 pages in 1.5 hours, how many pages does she read in 45 minutes?

35. Write a function that is $f(x) = \frac{1}{x}$ translated up 2 units and 6 units to the left.

36. Which function is a vertical translation and a vertical stretch of $f(x) = \frac{1}{x}$?

a. $f(x) = \frac{x}{2}$ **b.** $f(x) = \frac{1}{x + 2}$

c. $f(x) = \frac{2}{x} + 1$ **d.** $f(x) = \frac{1}{2} + x$

Use $f(x) = \frac{2}{x+1} - 5$ to answer Items 37–38.

37. What is the vertical asymptote of f?

a. $x = -5$ **b.** $x = -1$

c. $x = 1$ **d.** $x = 2$

38. What is vertical stretch of f?

a. 2 **b.** 1

c. -5 **d.** none

39. Graph the rational function $f(x) = \frac{-2}{x+5}$.

ACTIVITY 5.6

40. What are the restrictions on x in the rational expression $\frac{16 - x^2}{4x + 16}$?

a. none **b.** $x \neq 0$

c. $x \neq -4$ **d.** $x \neq \pm 4$

41. Simplify $\frac{16 - x^2}{4x + 16}$. Assume no denominator equals zero.

a. $4 - x$ **b.** $1 - \frac{x}{4}$

c. $\frac{4 - x}{4}$ **d.** $\frac{4 - x}{x + 4}$

42. Divide $\frac{4x + 4}{x^2} \div \frac{x^2 - 1}{x^2 - x}$. Assume no denominator is zero.

a. 4 **b.** $\frac{4}{x}$

c. $4 - x$ **d.** $\frac{4x + 4}{x}$

43. Simplify $\frac{6}{x - 6} + \frac{x}{x + 6}$.

a. -1 **b.** $\frac{x + 6}{x - 6}$

c. $\frac{1}{x - 6}$ **d.** $\frac{x^2 + 36}{(x - 6)(x + 6)}$

44. Simplify $\dfrac{\frac{1}{x-1} - \frac{1}{x}}{\frac{1}{x+1} - \frac{1}{x}}$. Assume no denominator equals 0.

a. 1 **b.** $\frac{1}{x}$

c. $1 - x$ **d.** $\frac{x + 1}{1 - x}$

45. Analyze and graph $f(x) = \frac{x^2 - 36}{x^2 - 5x - 6}$.

ACTIVITY 5.7

46. Solve $\frac{x + 3}{x - 6} = \frac{2}{5}$.

a. $x = -1$ **b.** $x = -9$

c. $x = 9$ **d.** no solution

47. Solve $\frac{2x}{x - 1} + \frac{x - 3}{x - 1} = 2$.

a. $x = 1$ **b.** $x = 0.5$

c. $x = 1, 0.5$ **d.** no solution

48. The resistance, R_P, of a circuit with 2 resistors in parallel is given in Ohms by $\frac{1}{R_P} = \frac{1}{R_1} + \frac{1}{R_2}$. If the resistors come in 5 ohm increments, find a pair of resistors so the circuit has a resistance of 6 ohms.

49. When Joe and Jon work together, they can wire a room in 8 hours. Working alone, Joe needs 12 hours to wire a room. How long would it take Jon working alone to wire a room?

50. Solve each inequality numerically or graphically.

a. $\frac{x + 3}{x^2 - 1} \leq 0$

b. $\frac{x^2 - x - 6}{x^2 + 2x - 3} > 0$

Reflection

An important aspect of growing as a learner is to take the time to reflect on your learning. It is important to think about where you started, what you have accomplished, what helped you learn, and how you will apply your new knowledge in the future. Use notebook paper to record your thinking on the following topics and to identify evidence of your learning.

Essential Questions

1. Review the mathematical concepts and your work in this unit before you write thoughtful responses to the questions below. Support your responses with specific examples from concepts and activities in the unit.
   - Why is it important to consider the domain and range of a function?
   - How are inverse functions useful in everyday life?

Academic Vocabulary

2. Look at the following academic vocabulary words:
   - complex fraction
   - horizontal asymptote
   - inverse variation
   - one-to-one function
   - power function
   - rational exponent
   - rational function
   - vertical asymptote

   Choose three words and explain your understanding of each word and why each is important in your study of math.

Self-Evaluation

3. Look through the activities and Embedded Assessments in this unit. Use a table similar to the one below to list three major concepts in this unit and to rate your understanding of each.

Unit Concepts	Is Your Understanding Strong (S) or Weak (W)?
Concept 1	
Concept 2	
Concept 3	

   a. What will you do to address each weakness?
   b. What strategies or class activities were particularly helpful in learning the concepts you identified as strengths? Give examples to explain.

4. How do the concepts you learned in this unit relate to other math concepts and to the use of mathematics in the real world?

1. Consider the functions: $c(x) = \dfrac{1}{\sqrt{x-4}}$ and $d(x) = 2x$. What is the domain of $c(d(x))$ over the set of real numbers?

A. $\{x: x \in \text{real numbers}\}$

B. $\{x: x > 4\}$

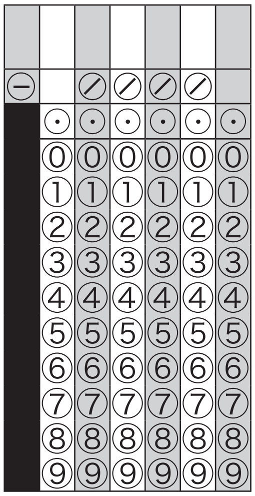
C. $\{x: x > 2\}$

D. $\{x: x > 0\}$

1. Ⓐ Ⓑ Ⓒ Ⓓ

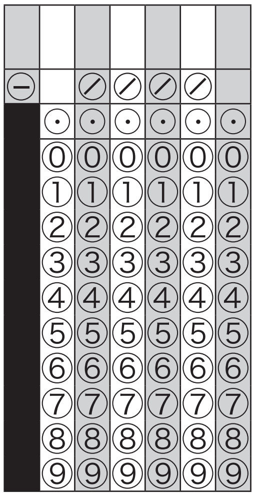
2. What is the solution of $x - 4 = \sqrt{2x}$?

2.



3. What is $(-27)^{\frac{4}{3}}$ in simplified form?

3.



Read
Solve
Explain

4. Given $f(x) = \sqrt{x-1} + 2$ and $g(x) = \sqrt{x}$.

Part A: Describe how to obtain the graph of $f(x)$ from the graph of $g(x)$.

Answer and Explain

Part B: State the domain and range of $f(x)$.

Answer and Explain

Part C: Graph $f(x)$ and $g(x)$ on the grid provided.