Unit 4

Polynomials

Unit Overview

In this unit you will study polynomials, beginning with operations and factoring and then investigating intercepts, end behavior, and relative maximums. You will also study permutations, combinations, and binomial probability.

Unit 4 Academic Vocabulary

Add these words and others you encounter in this unit to your vocabulary notebook.

- combination
- end behavior
- extrema
- factorial
- permutation
- polynomial function
- probability distribution

Essential Questions

How do polynomial functions help to model real-world behavior?

How is probability used in real-world settings?

EMBEDDED ASSESSMENTS

This unit has three embedded assessments, following Activities 4.2, 4.4, and 4.7. The first two will give you an opportunity to demonstrate your understanding of polynomial functions and the third assessment focuses on your understanding of permutations and combinations.

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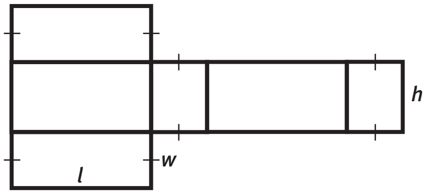
UNIT 4

Getting Ready

Write your answers on notebook paper or grid paper. Show your work.

1. Find the surface area and volume of a rectangular prism formed by the net below. The length is 10 units, the width is 4 units, and the height is 5 units.



2. Simplify $(2x^2 + 3x + 7) - (4x - 2x^2 + 9)$.

3. Factor $9x^4 - 49x^2y^2$.

4. Factor $2x^2 - 9x - 5$.

5. Two number cubes are tossed at the same time. Find each probability.

a. Exactly one cube shows a 3.

b. Both cubes show a 3.

c. Neither cube shows a 3.

6. A game spinner is 25% red, 25% blue, and 50% yellow. Draw a spinner that matches that description.

7. Using the spinner described in Item 6, find each probability.

a. spinning once and not landing on blue

b. spinning twice and landing on red both times

8. Find the sum and product of $(2 - 3i)$ and $(4 + 6i)$.

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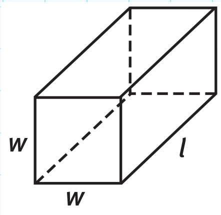
Introduction to Polynomials
Postal Service

ACTIVITY
4.1

SUGGESTED LEARNING STRATEGIES: Shared Reading, Create Representations, Think/Pair/Share

My Notes

The United States Postal Service will not accept rectangular packages if the perimeter of one end of the package plus the length of the package is greater than 130 in. Consider a rectangular package with square ends as shown in the figure.



1. Assume that the perimeter of one end of the package plus the length of the package equals the maximum 130 in. Complete the table with some possible measurements for the length and width of the package. Then find the corresponding volume of each package.

Width (in.)	Length (in.)	Volume (in.^{3})

2. Give an estimate for the largest possible volume of an acceptable United States Postal Service rectangular package with square ends.

CONNECT TO AP

In calculus, you must be able to model a written description of a physical situation with a function.

3. Use the package described in Item 1.

a. Write an expression for l, the length of the package, in terms of w, the width of the square ends of the package.

b. Write the volume of the package V as a function of w, the width of the square ends of the package.

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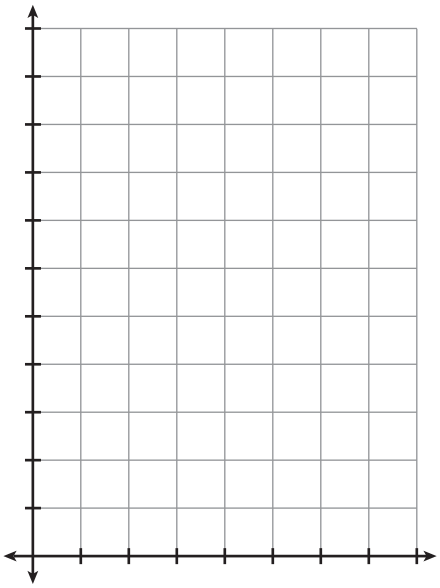
SUGGESTED LEARNING STRATEGIES: Create Representations, Note-taking, Quickwrite

My Notes

4. Consider the smallest and largest possible values for *w* that makes sense for the function you wrote in Item 3b. Give the domain of the function as a model of the volume of the postal package.

5. Sketch a graph of the function in Item 3(b) over the domain that you found in Item 4. Include the scale on each axis.



6. Use a graphing calculator to find the coordinates of the maximum point of the function that you graphed in Item 5.

7. What information do the coordinates of the maximum point of the function found in Item 6 provide with respect to an acceptable United States Postal Service package with square ends?

TECHNOLOGY TIP

Graphing calculators will allow you to find the maximum and minimum of functions in the graphing window.

CONNECT TO AP

In calculus, you will learn about the derivative of a function, which can be used to find the maximum and minimum values of a function.

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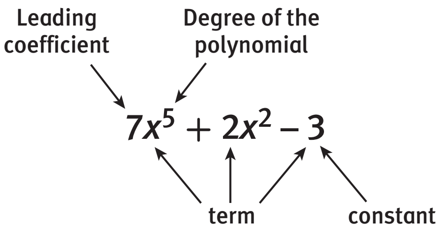
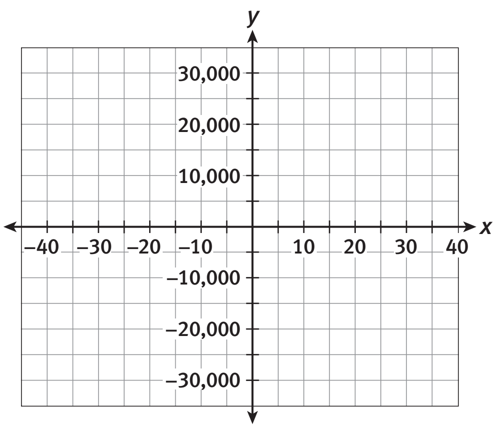
SUGGESTED LEARNING STRATEGIES: Marking the Text, Note-taking, Vocabulary Organizer, Interactive Word Wall, Create Representations

When using a function to model a given situation, such as the acceptable United States Postal Service package, you may be looking at only a portion of the entire domain of the function. Moving beyond the specific situation, you can examine the entire domain of the *polynomial function*.

A **polynomial function** in one variable is a function that can be written in the form $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$, where n is a nonnegative integer, the coefficients $a_0, a_1, \ldots a_n$ are real numbers, and $a_n \neq 0$. n is the **degree of the polynomial function**.



8. Write a polynomial function $f(x)$ defined over the set of real numbers such that it has the same function rule as $V(w)$ the rule you found in Item 3b. Sketch a graph of the function.



My Notes

ACADEMIC VOCABULARY

polynomial function

MATH TERMS

Polynomial functions are named by their **degree**. Here is a list of some common polynomial functions.

Degree	Name
0	Constant
1	Linear
2	Quadratic
3	Cubic
4	Quartic

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SUGGESTED LEARNING STRATEGIES: Vocabulary Organizer, Interactive Word Wall, Think/Pair/Share, Create Representations, Discussion Group

My Notes

MATH TERMS

A function value $f(a)$ is called a **relative maximum** of f if there is an interval around a where for any x in that interval $f(a) \geq f(x)$. A function value $f(a)$ is called a **relative minimum** of f if there is an interval around a where for any x in that interval $f(a) \leq f(x)$.

ACADEMIC VOCABULARY

end behavior

MATH TIP

Recall that the phrase "approaches positive infinity ∞" means "increases without bound," and that "approaches negative infinity $-\infty$" means "decreases without bound."

9. Name any **relative maximum** values and **relative minimum** values of the function $f(x)$ in Item 8.

10. Name any x- or y-intercepts of the function $f(x) = -4x^3 + 130x^2$.

When looking at the **end behavior** of a graph, you determine what happens to the graph on the extreme right and left ends of the x-axis. That is, you look to see what happens to y as x approaches $-\infty$ and ∞.

11. Examine the end behavior of $f(x) = -4x^3 + 130x^2$.

a. As x goes to ∞, what behavior does the function have?

b. How is the function behaving as x approaches $-\infty$?

12. Examine the end behavior of $f(x) = 3x^2 - 6$.

a. As x goes to ∞, what behavior does the function have?

b. How is the function behaving as x approaches $-\infty$?

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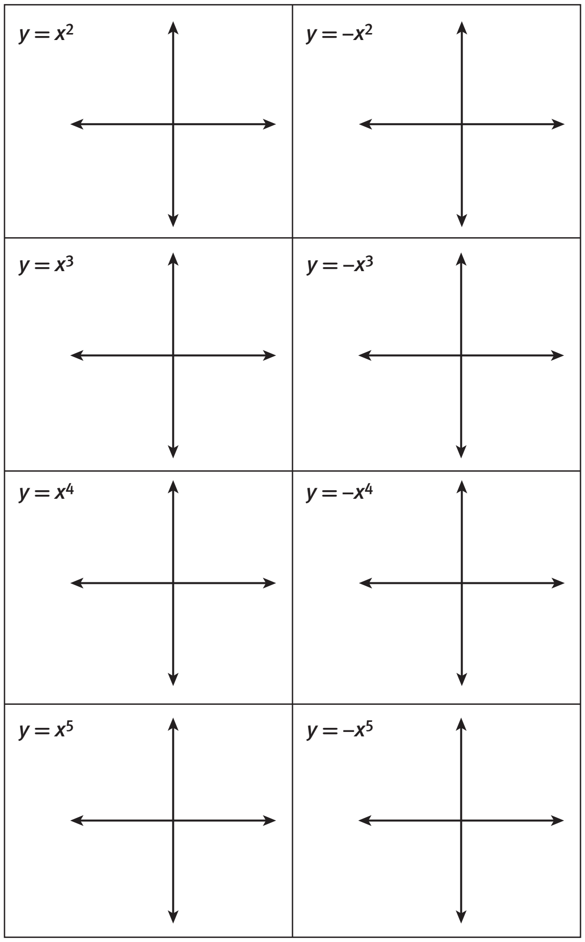
SUGGESTED LEARNING STRATEGIES: Create Representations, Think/Pair/Share

My Notes

13. Use a graphing calculator to examine the *end behavior* of polynomial functions in general. Sketch each given function on the axes below.



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SUGGESTED LEARNING STRATEGIES: Quickwrite, Group Presentation

My Notes

14. Make a conjecture about the end behavior of polynomial functions. Explain your reasoning.

CHECK YOUR UNDERSTANDING

Write your answers on notebook paper. Show your work.

For Items 1–4, decide if each function is a polynomial. If it is, write the function in standard form, and then state the degree and leading coefficient.

1. $f(x) = 5x - x^3 + 3x^5 - 2$

2. $f(x) = -\frac{2}{3}x^3 - 8x^4 - 2x + 7$

3. $f(x) = 4^x + 2x^2 + x + 5$

4. $f(x) = -5x^3 + x^6 + \frac{2}{x}$

5. Given $f(x) = 3x^3 + 5x^2 + 4x + 3$, find $f(3)$.

Describe the end behavior of each function.

6. $f(x) = x^6 - 2x^3 + 3x^2 + 2$

7. $f(x) = -x^3 + 7x^2 - 11$

8. MATHEMATICAL REFLECTION Which new concept in this investigation has been easiest for you to understand? Which one has been most difficult?

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Polynomial Operations
Polly's Pasta

ACTIVITY
4.2

SUGGESTED LEARNING STRATEGIES: Close Reading, Discussion Group, Create Representations, Think/Pair/Share, Self/Peer Revision

My Notes

Polly's Pasta and Pizza Supply sells wholesale goods to local restaurants. They keep track of revenue from kitchen supplies and food products. The function K models revenue from kitchen supplies and the function F models revenue from food product sales for one year in dollars, where t represents the number of the month (1–12) on the last day of the month.

$$K(t) = 15t^3 - 312t^2 + 1600t + 1100$$
$$F(t) = 36t^3 - 720t^2 + 3800t - 1600$$

MATH TIP

Some companies run their business on a fiscal year from July to June. Others, like Polly's Pasta, start the business year in January, so $t = 1$ represents January.

1. What kind of functions are these revenue functions?

2. How much did Polly make from kitchen supplies in March? How much did she make from selling food products in August?

3. In which month was her revenue from kitchen supplies the greatest? The least?

4. In which month was her revenue from food products the greatest? The least?

5. What was her total revenue from both kitchen supplies and food products in January? Explain how you arrived at you answer.

6. Complete the table for each given value of t.

t	$K(t)$	$F(t)$	$S(t) = K(t) + F(t)$
1			
2			
3			
4			
5			

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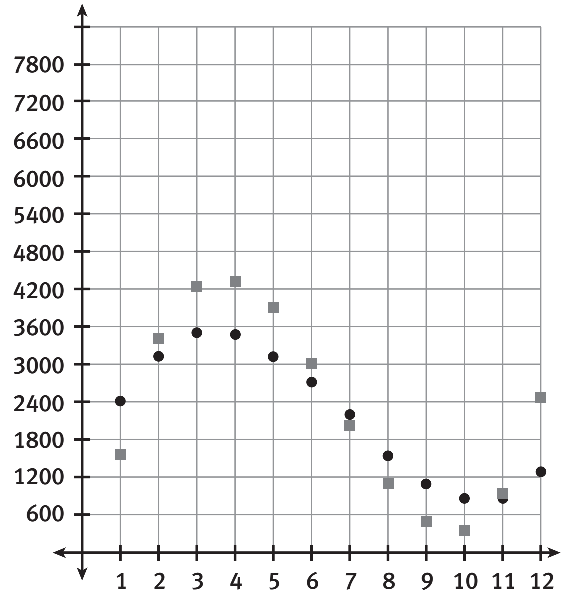
SUGGESTED LEARNING STRATEGIES: Create Representations, Quickwrite, Group Presentation

My Notes

7. The graph below shows $K(t)$ and $F(t)$. Graph $S(t) = K(t) + F(t)$, and explain how you used the graph to find the values of $S(t)$.



Patty's monthly costs are represented by the function

$$C(t) = 5t^3 - 110t^2 + 600t + 1000.$$

8. Profit equals total revenue minus total costs. How much profit did Patty earn in December? Explain how you found your solution.

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SUGGESTED LEARNING STRATEGIES: Create Representations, Note-taking

My Notes

9. Complete the table for each value of t.

t	$S(t)$	$C(t)$	$P(t) = S(t) - C(t)$
8			
9			
10			
11			
12			

To add and subtract polynomials, add or subtract the coefficients of like terms.

EXAMPLE 1

a. Add $(3x^3 + 2x^2 - 5x + 7) + (4x^2 + 2x - 3)$.

Step 1: *Group like terms* $(3x^3) + (2x^2 + 4x^2) + (-5x + 2x) + (7 - 3)$

Step 2: *Combine like terms.* $3x^3 + 6x^2 - 3x + 4$

Solution: $3x^3 + 6x^2 - 3x + 4$

b. Subtract $(2x^3 + 8x^2 + x + 10) - (5x^2 - 4x + 6)$.

Step 1: *Distribute the negative.* $2x^3 + 8x^2 + x + 10 - 5x^2 + 4x - 6$

Step 2: *Group like terms.* $2x^3 + (8x^2 - 5x^2) + (x + 4x) + (10 - 6)$

Step 3: *Combine like terms.* $2x^3 + 3x^2 + 5x + 4$

Solution: $2x^3 + 3x^2 + 5x + 4$

TRY THESE A

Find each sum or difference. Write your answers in the My Notes space. Show your work.

a. $(2x^4 - 3x + 8) + (3x^3 + 5x^2 - 2x + 7)$

b. $(4x - 2x^3 + 7 - 9x^2) + (8x^2 - 6x - 7)$

c. $(3x^2 + 8x^3 - 9x) - (2x^3 + 3x - 4x^2 - 1)$

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SUGGESTED LEARNING STRATEGIES: Think/Pair/Share, Activating Prior Knowledge, Quickwrite

My Notes

10. The **standard form of a polynomial** is $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$, where a is a real number and $a_n \neq 0$. Use what you learned about how to add and subtract polynomials to write $S(t)$ from Item 6 and $P(t)$ from Item 8 in standard form.

11. The steps to multiply $(x + 3)(4x^2 + 6x + 7)$ are shown below. Use appropriate math terminology to describe what occurs in each step.

$x(4x^2 + 6x + 7) + 3(4x^2 + 6x + 7)$	
$(4x^3 + 6x^2 + 7x) + (12x^2 + 18x + 21)$	
$4x^3 + 6x^2 + 12x^2 + 18x + 7x + 21$	
$4x^3 + 18x^2 + 25x + 21$	

TRY THESE **B**

Find each product. Write your answers in the My Notes space. Show your work.

a. $(x + 5)(x^2 + 4x - 5)$

b. $(2x^2 + 3x - 8)(2x - 3)$

c. $(x^2 - x + 2)(x^2 + 3x - 1)$

d. $(x^2 - 1)(x^3 + 4x)$

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12. When multiplying polynomials, how are the degrees of the factors related to the degree of the product?

SUGGESTED LEARNING STRATEGIES: Activating Prior Knowledge, Note-taking

My Notes

Polynomial long division has a similar algorithm to numerical long division.

13. Use long division to find the quotient $\frac{592}{46}$. Write your answer in the My Notes space.

EXAMPLE 2

Divide $x^3 - 7x^2 + 14$ by $x - 5$, using long division.

Step 1: *Set up the division problem with the divisor and dividend written in descending order of degree. Include zero coefficients for any missing terms.*

$$x - 5\overline{)x^3 - 7x^2 + 0x + 14}$$

Step 2: *Divide the first term of the dividend* [x^3] *by the first term of the divisor* [x].

$$\begin{array}{r} x^2 \\ x - 5\overline{)x^3 - 7x^2 + 0x + 14} \end{array}$$

Step 3: *Multiply the result* [x^2] *by the divisor* [$x^2(x - 5) = x^3 - 5x^2$].

$$\begin{array}{r} x^2 \\ x - 5\overline{)x^3 - 7x^2 + 0x + 14} \\ x^3 - 5x^2 \end{array}$$

Step 4: *Subtract to get a new result* [$-2x^2 + 0x + 14$].

$$\begin{array}{r} x^2 \\ x - 5\overline{)x^3 - 7x^2 + 0x + 14} \\ \underline{-(x^3 - 5x^2)} \\ -2x^2 + 0x + 14 \end{array}$$

Step 5: *Repeat the steps.*

$$\begin{array}{r} x^2 - 2x - 10 \\ x - 5\overline{)x^3 - 7x^2 + 0x + 14} \\ \underline{-(x^3 - 5x^2)} \\ -2x^2 + 0x + 14 \\ \underline{-(-2x^2 + 10x)} \\ -10x + 14 \\ \underline{-(-10x + 50)} \\ -36 \end{array}$$

Solution: $\frac{x^3 - 7x^2 + 14}{x - 5} = x^2 - 2x - 10 - \frac{36}{x - 5}$

MATH TIP

When the division process is complete, the degree of the remainder will be less than the degree of the divisor.

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SUGGESTED LEARNING STRATEGIES: Create Representations

My Notes

When a polynomial function $f(x)$ is divided by another polynomial function $d(x)$, the outcome is a new quotient function consisting of a polynomial $p(x)$ plus a remainder function $r(x)$.

$$\frac{f(x)}{d(x)} = p(x) + \frac{r(x)}{d(x)}$$

14. Follow the steps from Example 2 to find the quotient of $\frac{x^3 - x^2 + 4x + 6}{x + 2}$.

$$x + 2\overline{)x^3 - x^2 + 4x + 6}$$

TRY THESE **C**

Use long division to find each quotient. Write your answers in the My Notes space. Show your work.

a. $(x^2 + 5x - 3) \div (x - 5)$

b. $\frac{4x^4 + 12x^3 + 7x^2 + x + 6}{-2x + 3}$

c. $\frac{-4x^3 - 8x^2 + 32x}{x^2 + 2x - 8}$

d. $(x^3 - 9) \div (x + 3)$

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SUGGESTED LEARNING STRATEGIES: Note-taking, Discussion Group

My Notes

Synthetic division is another method of polynomial division that is useful when the divisor has the form $x - k$.

EXAMPLE 3

Divide $x^4 - 13x^2 + 32$ by $x - 3$.

Step 1: *Set up the division problem using only coefficients for the dividend and only the constant for the divisor. Include zero coefficients for any missing terms* [x^3 *and* x].

$3|\ 10 \quad -13 \quad 0 \quad 32$

Step 2: *Bring down the leading coefficient* [1].

3	1	0	−13	0	32
	↓				
	1				

Step 3: *Multiply the coefficient* [1] *by the divisor* [3]. *Write the product* [$1 \cdot 3 = 3$] *under the second coefficient* [0] *and add* [$0 + 3 = 3$].

3	1	0	−13	0	32
		3			
	1	3			

Step 4: *Repeat this process until there are no more coefficients.*

3	1	0	−13	0	32
		3	9	−12	−36
	1	3	−4	−12	−4

Step 5: *The numbers in the bottom row become the coefficients of the quotient. The number in the last column is the remainder. Write it over the divisor.*

$$x^3 + 3x^2 - 4x - 12 - \frac{4}{x - 3}$$

Solution: $x^3 + 3x^2 - 4x - 12 - \dfrac{4}{x - 3}$

MATH TIP

In synthetic division, the quotient is always one degree less than the dividend.

15. Use synthetic division to divide $\dfrac{x^3 - x^2 + 4x + 6}{x + 2}$.

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My Notes

TRY THESE **D**

Use synthetic division to find each quotient.

a. $\dfrac{x^3 + 3x^2 - 10x - 24}{x + 4}$

b. $\dfrac{-5x^5 - 2x^4 + 32x^2 - 48x + 32}{x - 2}$

CHECK YOUR UNDERSTANDING

Write your answers on notebook paper. Show your work.

Find each sum or difference.

1. $(3x^2 - 4) + (5x^2 + 1)$
2. $(x^2 - 6x + 5) - (2x^2 + x + 1)$
3. $(4x^2 - 12x + 9) + (3x - 11)$
4. $(6x^2 - 13x + 4) - (8x^2 - 7x + 25)$

Find each product.

5. $3x(x^2 + 7x + 8)$
6. $(x - 3)(2x^3 - 9x^2 + x - 6)$

Find each quotient, using long division.

7. $\dfrac{x^2 - 6x + 4}{x + 1}$
8. $(5x^4 + 14x^3 + 9x) \div (x^2 + 3x + 1)$

Find each quotient, using synthetic division.

9. $(x^2 + 4) \div (x + 4)$
10. $\dfrac{3x^3 - 10x^2 + 12x - 22}{x - 4}$
11. **MATHEMATICAL REFLECTION** Which operations on polynomials have been easy for you to understand? Which have been more difficult?

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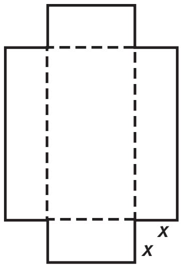
Polynomial Operations

THIS TEST IS SQUARE

Embedded Assessment 1

Use after Activity 4.2.

Congruent squares of length *x* are cut from the corners of a 10 in. by 15 in. piece of cardboard to create a box without a lid.



1. Write an expression for each:

a. height of the box

b. length of the box

c. width of the box

2. Write a function $V(x)$ for the volume of the box in terms of x.

3. What are the possible values of x?

4. Use a graphing calculator to determine the value of x that gives the maximum volume.

Use these functions for Items 5–8.

$f(x) = x + 4$ $\qquad$ $g(x) = -x^2 + 9x + 20$

$p(x) = x^2 - x - 6$ $\qquad$ $h(x) = x^3 + 11x^2 + 38x + 40$

5. Find $g(x) + p(x)$.

6. Find $f(x) \cdot p(x)$.

7. Find $\frac{p(x)}{f(x)}$, using long division.

8. Find $\frac{h(x)}{f(x)}$, using synthetic division.

9. Write a function of degree 5 or higher that has this end behavior:

- as x goes to ∞, y approaches ∞
- as x goes to $-\infty$, y approaches ∞

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Embedded Assessment 1

Use after Activity 4.2.

Polynomial Operations

THIS TEST IS SQUARE

	Exemplary	Proficient	Emerging
Math Knowledge #5, 6, 7, 8	The student: • Finds correct expressions for $g(x) + p(x)$ and $f(x) \times p(x)$. (5, 6) • Uses long division to find the correct quotient for $p(x)/f(x)$ and uses synthetic division to find the correct quotient for $h(x)/f(x)$. (7, 8)	The student: • Finds only one of the correct expressions. • Uses only one of the methods to find both correct quotients.	The student: • Finds neither of the correct expressions. • Attempts, unsuccessfully, to find the quotients.
Problem Solving #3, 4, 9	The student: • Gives the correct interval for the possible values of *x*. (3) • Determines the correct value of *x* that gives the maximum value. (4) • Writes a correct fifth or higher degree function with both of the given characteristics. (9)	The student: • Provides all of the correct integer values of *x*. • Writes a function of degree five or higher with one of the given characteristics.	The student: • Provides at least two correct integer values for *x*. • Does not determine the correct value of *x*. • Writes a function of degree five or higher with neither of the given characteristics.
Representations #1a, b, c; 2	The student: • Writes correct expressions for the height, length, and width of the box. (1a, b, c) • Writes a correct function for the volume of the box. (2)	The student: • Writes correct expressions for only two of the dimensions of the box. • Writes a function for the volume of the box that is correct for the incorrect dimensions given in question 1	The student: • Writes a correct expression for only one of the dimensions of the box. • Writes an incorrect function.

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Factors of Polynomials
Factoring For Experts

ACTIVITY 4.3

SUGGESTED LEARNING STRATEGIES: Shared Reading, Activating Prior Knowledge, Discussion Group, Note-taking

My Notes

When you factor a polynomial, you rewrite the original polynomial as a product of two or more polynomial factors.

1. State the common factor of the terms in the polynomial $4x^3 + 2x^2 - 6x$. Then factor the polynomial.

2. Consider the expression $x^2(x - 3) + 2x(x - 3) + 3(x - 3)$.

a. How many terms does it have?

b. What factor do all the terms have in common?

3. Factor $x^2(x - 3) + 2x(x - 3) + 3(x - 3)$.

Some quadratic trinomials, $ax^2 + bx + c$, can be factored into two binomial factors.

EXAMPLE 1

Factor $2x^2 + 7x - 4$.

Step 1:	*Find the product of* a *and* c.	$2(-4) = 8$
Step 2:	*Find the factors of* ac *that have a sum of* b, 7.	$8 + (-1) = 7$
Step 3:	*Rewrite the polynomial, separating the linear term.*	$2x^2 + 8x - 1x - 4$
Step 4:	*Group the first two terms and the last two terms.*	$(2x^2 + 8x) + (-x - 4)$
Step 5:	*Factor each group separately.*	$2x(x + 4) - 1(x + 4)$
Step 6:	*Factor out the binomial.*	$(x + 4)(2x - 1)$

Solution: $(x + 4)(2x - 1)$

MATH TIP

Check your answer to a factoring problem by multiplying the factors together to get the original polynomial.

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SUGGESTED LEARNING STRATEGIES: Activating Prior Knowledge, Note-taking

My Notes

TRY THESE A

a. Use Example 1 as a guide to factor $6x^2 + 19x + 10$.

Factor each trinomial. Write your answers in the My Notes space. Show your work.

b. $3x^2 - 8x - 3$

c. $2x^2 + 7x + 6$

Some higher-degree polynomials can also be *factored by grouping.*

EXAMPLE 2

a. Factor $3x^2 + 9x^2 + 4x + 12$ by grouping.

Step 1: *Group the terms.* $(3x^3 + 9x^2) + (4x + 12)$

Step 2: *Factor each group separately.* $3x^2(x + 3) + 4(x + 3)$

Step 3: *Factor out the binomial.* $(x + 3)(3x^2 + 4)$

Solution: $(x + 3)(3x^2 + 4)$

b. Factor $3x^3 + 4x + 9x^2 + 12$ by grouping.

Step 1: *Group the terms.* $(3x^3 + 4x) + (9x^2 + 12)$

Step 2: *Factor each group separately.* $x(3x^2 + 4) + 3(3x^2 + 4)$

Step 3: *Factor out the binomial.* $(3x^2 + 4)(x + 3)$

Solution: $(3x^2 + 4)(x + 3)$

TRY THESE B

Factor by grouping. Write your answers in the My Notes space. Show your work.

a. $2x^3 + 10x^2 - 3x - 15$

b. $4x^3 + 3x^2 + 4x + 3$

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SUGGESTED LEARNING STRATEGIES: Marking the Text, Shared Reading, Look for a Pattern, Identify a Subtask, Simplify a Problem, Activating Prior Knowledge

My Notes

A difference of two squares can be factored by using a specific pattern, $a^2 - b^2 = (a + b)(a - b)$. A *difference of two cubes* and a *sum of two cubes* also have a factoring pattern.

Difference of Cubes	Sum of Cubes
$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$	$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

4. What patterns do you notice in the formulas that appear above?

TRY THESE C

Factor each difference or sum of cubes.

a. $x^3 - 8$

b. $x^3 + 27$

c. $8x^3 - 64$

d. $27 + 125x^3$

Some higher-degree polynomials can be factored by using the same patterns or formulas that you used when factoring quadratic binomials or trinomials.

5. Use the difference of squares formula $a^2 - b^2 = (a + b)(a - b)$ to factor $16x^4 - 25$. (It may help to write each term as a square.)

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My Notes

SUGGESTED LEARNING STRATEGIES: Quickwrite, Think/Pair/Share, Identify a Subtask, Simplify the Problem, Vocabulary Organizer, Note-taking

6. Explain the steps used to factor $2x^5 + 6x^3 - 8x$.

$2x^5 + 6x^3 - 8x$	Original expression
$= 2x(x^4 + 3x^2 - 4)$	
$= 2x(x^2 + 4)(x^2 - 1)$	
$= 2x(x^2 + 4)(x + 1)(x - 1)$	

TRY THESE D

Use the formulas for quadratic trinomials to factor each expression.

a. $x^4 + x^2 - 20$

b. $16x^4 - 81$

c. $(x - 2)^4 + 10(x - 2)^2 + 9$

MATH TERMS

Let $p(x)$ be a polynomial function of degree n, where $n > 0$. The **Fundamental Theorem of Algebra** states that $p(x) = 0$ has at least one zero in the complex number system.

As a consequence of the **Fundamental Theorem of Algebra**, a polynomial $p(x)$ of degree $n \geq 0$ has exactly n linear factors, counting multiple factors.

EXAMPLE 3

Find the zeros of $f(x) = 3x^3 + 2x^2 + 6x + 4$.

Step 1: Set the function equal to 0. $3x^3 + 2x^2 + 6x + 4 = 0$

Step 2: *Look for a factor common to all terms, use the quadratic trinomial formulas, or factor by grouping, as was done here.* $(3x^3 + 6x) + (2x^2 + 4) = 0$

Step 3: *Factor each group separately.* $3x(x^2 + 2) + 2(x^2 + 2) = 0$

Step 4: *Factor out the binomial to write the factors.* $(x^2 + 2)(3x + 2) = 0$

Step 5: *Use the Zero Product Property to solve for x.* $x^2 + 2 = 0$, $x = \pm i\sqrt{2}$; $3x + 2 = 0$, $x = -\frac{2}{3}$

Solution: $x = \pm i\sqrt{2}; x = -\frac{2}{3}$

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SUGGESTED LEARNING STRATEGIES: Graphic Organizer, Group Presentation, Vocabulary Organizer, Note-taking, Work Backward

My Notes

TRY THESE E

Find the zeros of the functions by factoring and using the Zero Product Property.

a. $f(x) = x^3 + 9x$

b. $g(x) = x^4 - 16$

c. $h(x) = (x - 2)^2 + 4(x - 2) + 4$

d. $k(x) = x^3 - 3x^2 - 15x + 125$

e. $p(x) = x^3 - 64$

f. $w(x) = x^3 + 216$

7. Create a flow chart, other organizational scheme, or set of directions for factoring polynomials.

It is possible to find a polynomial function, given its zeros. The **Complex Conjugate Root Theorem** states that if $a + bi$, $b \neq 0$, is a zero of a polynomial function with real coefficients, the conjugate $a - bi$ is also a zero of the function.

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EXAMPLE 4

Find a polynomial function of 4th degree that has zeros 1, −1, and $1 + 2i$.

Step 1: *Use the Complex Conjugate Root Theorem to find all zeros.* $x = 1, x = -1, x = 1 + 2i, x = 1 - 2i$

Step 2: *Write the factors.* $f(x) = (x - 1)(x + 1)(x - (1 + 2i))(x - (1 - 2i))$

Step 3: *Multiply using the fact that* $(a - b)(a + b) = a^2 - b^2$. $f(x) = (x^2 - 1)(x^2 - 2x + 5)$

Step 4: *Multiply out the factors to get the polynomial function.* $f(x) = x^4 - 2x^3 + 4x^2 + 2x - 5$

Solution: $f(x) = x^4 - 2x^3 + 4x^2 + 2x - 5$

SUGGESTED LEARNING STRATEGIES: Work Backward

My Notes

TRY THESE F

Write a polynomial function of nth degree that has the given real or complex roots. Write your answers on a separate sheet of notebook paper. Show your work.

a. $n = 3;\ x = -2,\ x = 3i$

b. $n = 4;\ x = 3,\ x = -3,\ x = 1 + 2i$

c. $n = 4;\ x = 2,\ x = -5$, and $x = -4$ is a double root

CHECK YOUR UNDERSTANDING

Write your answers on notebook paper. Show your work.

1. Factor by grouping.

   a. $8x^3 - 64x^2 + x - 8$

   b. $12x^3 + 2x^2 - 30x - 5$

2. Factor each difference or sum of cubes.

   a. $125x^3 + 216$

   b. $x^6 - 27$

3. Use the formulas for factoring quadratic trinomials to factor each expression.

   a. $x^4 - 14x^2 + 33$

   b. $81x^4 - 625$

   c. $x^4 + 17x^2 + 60$

4. Find the zeros of the functions by factoring and using the Zero Product Property.

   a. $f(x) = 2x^4 + 18x^2$

   b. $g(x) = 3x^3 - 3$

   c. $h(x) = 5x^3 - 6x^2 - 45x + 54$

5. The table of values shows coordinate pairs on the graph of $f(x)$. Which of the following could be $f(x)$?

   a. $x(x + 1)(x - 1)$

   b. $(x - 1)(x + 1)(x - 3)$

   c. $(x + 1)^2(x + 3)$

   d. $(x + 1)(x - 2)^2$

   e. $x(x - 1)(x + 3)$

x	$f(x)$
−1	0
0	3
1	0
2	−3

6. Write a polynomial function of nth degree that has the given real or complex roots.

   a. $n = 3;\quad x = -2,\ x = 5,\ x = -5$

   b. $n = 4;\quad x = -3,\ x = 3,\ x = 5i$

7. **MATHEMATICAL REFLECTION** How do you think memorizing the factoring patterns for the sum and difference of cubes and a difference of squares will benefit you as you progress in mathematics?

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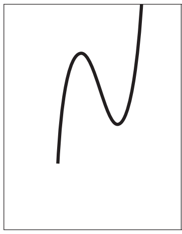
Graphs of Polynomials
Graphing Polynomials

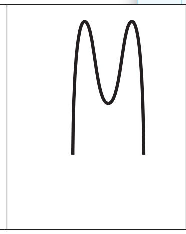
ACTIVITY
4.4

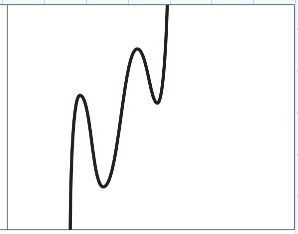
SUGGESTED LEARNING STRATEGIES: Quickwrite, Look for a Pattern, Group Presentation, Create Representations, Think/Pair/Share

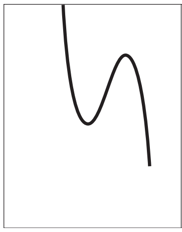
My Notes

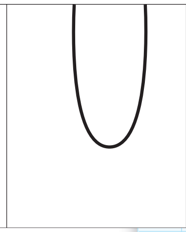
1. Each graph to the right shows a polynomial of the form $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$, where $a_n \neq 0$. Use each graph to make a conjecture about how the leading coefficient and degree affect the end behavior of the function.

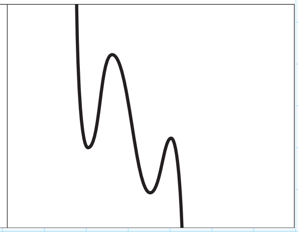

$y = 2x^3 - 4x^2 + 1$


$y = -3x^4 + 8x^2 + 1$


$y = 2x^5 + 4x^4 - 5x^3 - 8x^2 + 5x$


$y = -2x^3 - 4x^2 + 1$


$y = 3x^4 - 8x^2 + 1$


$y = -2x^5 - 4x^4 + 5x^3 + 8x^2 - 5x$

2. Use what you know about end behavior and zeros of a function to sketch a graph of each function in the My Notes section.

a. $f(x) = x + 3$

b. $g(x) = x^2 - 9 = (x + 3)(x - 3)$

c. $h(x) = x^3 + x^2 - 9x - 9 =$
$(x + 3)(x - 3)(x + 1)$

d. $k(x) = x^4 - 10x^2 + 9 =$
$(x + 3)(x - 3)(x + 1)(x - 1)$

e. $p(x) = x^5 + 10x^4 + 37x^3 + 60x^2 + 36x =$
$x(x + 2)^2(x + 3)^2$

MATH TIP

If $(x - a)$ is a factor of a polynomial $f(x)$, then a is an x-intercept of the graph $f(x)$.

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Graphs of Polynomials
Graphing Polynomials

SUGGESTED LEARNING STRATEGIES: Vocabulary Organizer, Marking the Text, Question the Text, Create Representations, Think/Pair/Share

My Notes

ACADEMIC VOCABULARY

Maxima and *minima* are known as **extrema**. They are the greatest value (the maximum) or the least value (the minimum) of a function. When these values occur at a point within a given interval, they are called *relative extrema*. When they occur on the entire domain of the function, they are called *global extrema*.

CONNECT TO AP

In calculus, you will use the first derivative of a polynomial function to algebraically determine the coordinates of the extrema.

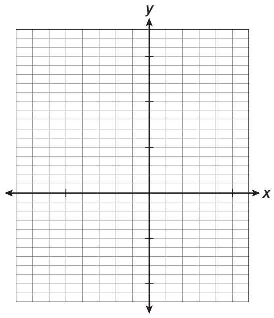
Polynomial functions are **continuous functions**, meaning that their graphs have no gaps or breaks. Their graphs are smooth, unbroken curves with no sharp turns. Graphs of polynomial functions with degree n have n zeros, as you saw in the Fundamental Theorem of Algebra. They also have at most $n - 1$ **relative extrema** (*maximum* or *minimum* points).

3. Find the x-intercepts of $f(x) = x^4 + 3x^3 - x^2 - 3x$.

4. Find the y-intercept of $f(x)$.

5. How can the zeros of a polynomial function help you identify where the relative extrema will occur?

6. The relative extrema occur at approximately $x = 0.6$, $x = -0.5$, and $x = -2.3$. Find the approximate values of the extrema and graph $f(x) = x^4 + 3x^3 - x^2 - 3x$.



7. Sketch a graph of $f(x) = -x^3 \ -x^2 - 6x$ in the My Notes section.

8. Sketch a graph of $f(x) = x^4 - 10x^2 + 9$ below.

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SUGGESTED LEARNING STRATEGIES: Shared Reading, Vocabulary Organizer, Predict and Confirm, Quickwrite

My Notes

The function $f(x) = x^3 - 2x^2 - 5x + 6$ is not factorable using the tools that you have. However, to graph a function of this form without a calculator, the following tools will be helpful.

The Rational Root Theorem	Finds possible rational roots.
Descartes' Rule of Signs	Finds the possible number of real roots.
The Remainder Theorem	Determines if a value is a zero.
The Factor Theorem	Another way to determine if a value is a zero.

The Rational Root Theorem

If a polynomial function $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$, $a_n \neq 0$, has integer coefficients, then every rational root of $f(x) = 0$ has the form $\frac{p}{q}$, where p is a factor of a_0, and q is a factor of a_n.

The Rational Root Theorem determines the possible rational roots of the polynomial.

9. Consider the quadratic equation $2x^2 + 9x - 3 = 0$.

a. Make a list of the only possible rational roots to this equation.

b. Explain why you think these are the only possible rational roots.

c. Does your list of rational roots satisfy the equation?

d. What can you conclude from Part c?

e. Verify your conclusion in Part c by finding the roots of the quadratic by using the Quadratic Formula.

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SUGGESTED LEARNING STRATEGIES: Predict and Confirm, Quickwrite, Note-taking, Shared Reading, Vocabulary Organizer

My Notes

EXAMPLE 1

Find all the possible rational zeros of $f(x) = x^3 - 2x^2 - 5x + 6$.

Step 1: *Find the factors* q *of the leading coefficient 1 and the factors* p *of the constant term 6.*

q could equal ±1

p could equal $\pm1, \pm2, \pm3, \pm6$

Step 2: *Write all combinations of* $\frac{p}{q}$. *Then simplify.*

$\frac{\pm1, \pm2, \pm3, \pm6}{\pm1}$

Solution: $\pm1, \pm2, \pm3, \pm6$

The Rational Root Theorem can yield a large number of possible roots. To help eliminate some possibilities, you can use Descartes' Rule of Signs. While Descartes' rule does not tell you the value of the roots, it does tell you the maximum number of positive and negative real roots.

Descartes' Rule of Signs

If $f(x)$ is a polynomial function with real coefficients and a nonzero constant term arranged in descending powers of the variable, then

- the number of positive real roots of $f(x) = 0$ equals the number of variations in sign of the terms of $f(x)$, or is less than this number by an even integer.
- the number of negative real roots of $f(x) = 0$ equals the number of variations in sign of the terms of $f(-x)$, or is less than this number by an even integer.

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SUGGESTED LEARNING STRATEGIES: Note-taking, Marking the Text, Vocabulary Organizer

My Notes

EXAMPLE 2

Find the number of positive and negative roots of $f(x) = x^3 - 2x^2 - 5x + 6$.

Step 1: *Determine the sign changes in* $f(x)$: $f(x) = x^3 - 2x^2 - 5x + 6$

There are 2 sign changes:

- one between the 1st and 2nd terms when the sign goes from positive to negative
- one between the 3rd and 4th terms when the sign goes from negative to positive

So there are either 2 or 0 positive real roots.

Step 2: *Determine the sign changes in* $f(-x)$: $f(-x) = -x^3 - 2x^2 + 5x + 6$

There is 1 sign change:

- between the 2nd and the 3rd terms when the sign goes from negative to positive

So there is 1 negative real root.

Solution: There are either 2 or 0 positive real roots and 1 negative real root.

You have found all the possible rational roots and the number of positive and negative real roots of a polynomial. The theorems below help you to find the zeros of the function. The Remainder Theorem tells if the factor is a zero, or another point on the polynomial. The Factor Theorem gives another way to test if a possible root is a zero.

The Remainder Theorem

If a polynomial $P(x)$ is divided by $(x - k)$ where k is a constant, then the remainder r is $P(k)$.

The Factor Theorem

A polynomial $P(x)$ has a factor $(x - k)$ if and only if $P(k) = 0$.

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SUGGESTED LEARNING STRATEGIES: Note-taking

My Notes

EXAMPLE 3

Use synthetic division to find the zeros and factor $f(x) = x^3 - 2x^2 - 5x + 6$.

From Examples 1 and 2, you know the possible rational zeros are ±1, ±2, ±3, ±6. You also know that the polynomial has either 2 or 0 positive real roots and 1 negative real root.

Step 1: *Divide* $(x^3 - 2x^2 - 5x + 6)$ *by* $(x + 1)$.

$$\begin{array}{r|rrrr} -1 & 1 & -2 & -5 & 6 \\ & & -1 & 3 & 2 \\ \hline & 1 & -3 & -2 & 8 \end{array}$$

So you have found a point (−1, 8).

Step 2: *Continue this process, finding either points on the polynomial and/or zeros for each of the possible roots.*

Divide $(x^3 - 2x^2 - 5x + 6)$ by $(x - 1)$

$$\begin{array}{r|rrrr} 1 & 1 & -2 & -5 & 6 \\ & & 1 & -1 & -6 \\ \hline & 1 & -1 & -6 & 0 \end{array}$$

So you have found a point (1, 0) and a factor, $f(x) = (x - 1)(x^2 - x - 6)$.

Step 3: *As soon as you have a quadratic factor remaining after the division process, you can factor the quadratic factor by inspection, if possible, or use the Quadratic Formula.*

Solution: $f(x) = (x - 1)(x + 2)(x - 3)$; the real zeros are 1, −2, and 3.

Using the Factor Theorem, follow a similar process to find the real zeros.

EXAMPLE 4

Use the Factor Theorem to find the real zeros of $f(x) = x^3 - 2x^2 - 5x + 6$. Again, you know the possible rational roots are ±1, ±2, ±3, ±6.

Step 1: *Test* $(x + 1)$: $f(-1) = (-1)^3 - 2(-1)^2 - 5(-1) + 6 = 8$
So you have a point (−1, 8).

Step 2: *Test* $(x - 1)$: $f(1) = (1)^3 - 2(1)^2 - 5(1) + 6 = 0$
So you have a zero at $x = 1$.

Step 3: *Test* $(x - 2)$: $f(2) = (2)^3 - 2(2)^2 - 5(2) + 6 = -4$

Step 4: *Continue to test rational zeros or use division to simplify the polynomial and factor or use the quadratic formula to find the real zeros.*

Solution: The real zeros are 1, −2, and 3.

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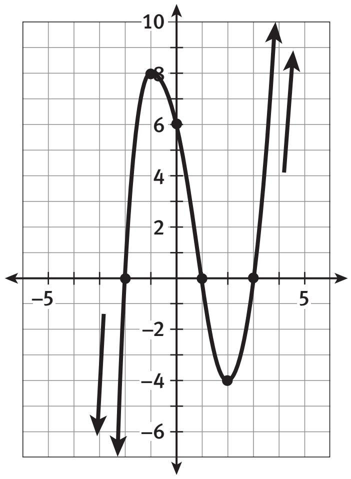
SUGGESTED LEARNING STRATEGIES: Note-taking, Create Representations

My Notes

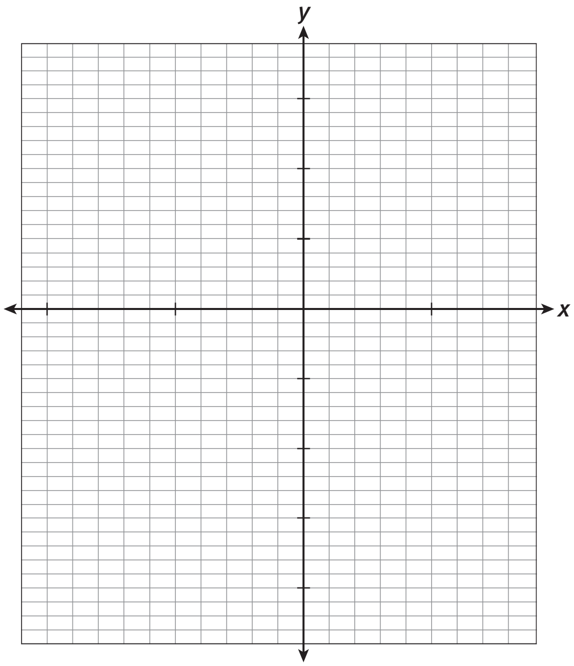
EXAMPLE 5

Graph $f(x) = x^3 - 2x^2 - 5x + 6$, using the information you have found so far, including the y-intercept and the end behavior of the function.

x	y
−1	8
1	0
−2	0
0	6
3	0
2	−4



10. Follow the examples above to graph $f(x) = 2x^3 + 7x^2 + 2x - 3$.



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SUGGESTED LEARNING STRATEGIES: Note-taking, Simplify the Problem, Identify a Subtask, Create Representations

My Notes

To solve a **polynomial inequality** by graphing, use the fact that a polynomial can only change signs at its zeros.

Step 1: Write the polynomial inequality with one side equal to zero.

Step 2: Graph the inequality and determine the zeros.

Step 3: Find the intervals where the conditions of the inequality are met.

11. Solve the polynomial inequality $x^4 - 13x^2 + 6 < -30$ by graphing on a graphing calculator or by hand.

CHECK YOUR UNDERSTANDING

Write your answers on notebook paper or grid paper. Show your work.

Determine the end behavior of each function.

1. $y = -3x^5 - 4x^3 + 5x + 7$

2. $y = 5x^{12} + 43x^8 - 14x^5 + 12x^2 + 8x$

Use what you know about end behavior and zeros to graph each function.

3. $y = x^5 - 2x^4 - 25x^3 + 26x^2 + 120x$
$= x(x - 5)(x - 3)(x + 2)(x + 4)$

4. $y = x^5 + 9x^4 + 16x^3 - 60x^2 - 224x - 192$
$= (x - 3)(x + 2)^2(x + 4)^2$

5. Determine all the possible rational zeros of $f(x) = x^3 - 2x^2 - 4x + 5$.

6. Graph $f(x) = x^3 - 2x^2 - 4x + 5$.

7. Determine the possible number of positive and negative real zeros for $h(x) = x^3 - 4x^2 + x + 5$.

8. Graph $h(x) = x^3 - 4x^2 + x + 5$.

9. Solve the inequality $x^3 - 2x < 0$.

10. MATHEMATICAL REFLECTION Write a paragraph arguing for or against the use of graphing calculators in graphing and understanding polynomial functions.

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Factoring and Graphing Polynomials

SKETCH ARTIST

Embedded Assessment 2

Use after Activity 4.4.

1. Factor $f(x) = x^3 + 3x^2 - x - 3$. Then find the zeros and y-intercept. Sketch a graph of the function.

2. Find two different ways to show that $g(x) = -x^3 + 27$ has only one x-intercept. Use a sketch of the graph as one method, if necessary.

3. List all the characteristics of the graph for this polynomial function that you would expect to see, based on what you have learned thus far.

$$f(x) = (x + 3)(x - 3)(x + 2)(x - 1)(x + 2i)(x - 2i)$$

4. Find a polynomial function of 4th degree that has the zeros 2, -2, and $1 - 3i$. Then write it in standard form.

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Embedded Assessment 2

Use after Activity 4.4.

Factoring and Graphing Polynomials

SKETCH ARTIST

	Exemplary	Proficient	Emerging
Math Knowledge #1	The student factors $f(x)$ correctly. (1)	The student writes only one correct factor.	The student writes no correct factors.
Problem Solving #1, 2, 3, 4	The student: • Finds correct values for the three zeros and the y-intercept of $f(x)$. (1) • Uses two different correct ways to show that $g(x)$ has only one x-intercept. (2) • Lists the correct zeros, y-intercept, degree, and end-behavior of $f(x)$. (3) • Writes a correct polynomial function in standard form. (4)	The student: • Finds only three of the correct values for the zeros and y-intercept. • Uses only one way to show that $g(x)$ has only one x-intercept. • Lists at least four of the characteristics of $f(x)$ correctly. • Writes a correct polynomial function, but not in standard form.	The student: • Finds only one or two of the correct values. • Is not successful in showing that $g(x)$ has only one x-intercept. • Lists at least two of the characteristics of $f(x)$ correctly. • Writes an incorrect 4^{th} degree polynomial.
Representations #1	The student sketches a correct graph of $f(x)$. (1)	The student sketches a partially correct graph of $f(x)$.	The student sketches a graph that has no correct features.

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Counting Methods
Let Me Count the Ways

ACTIVITY 4.5

SUGGESTED LEARNING STRATEGIES: Role Play, Graphic Organizer, Simplify the Problem

Sandwich Shop offers a combo meal that includes a choice of four sandwiches, three sides, and five drinks. The *Sandwich Shop* menu is shown at the right.

1. How many different combo meals consisting of one sandwich, one side dish, and one drink are offered at *Sandwich Shop*? Explain how you arrived at your answer.

The *Gold Diner* also offers a combo meal consisting of eight main dishes, four side dishes, and six drinks.

Gold Diner Menu		
Main Courses	**Side Dishes**	**Drinks**
Fiesta Chicken	Salad	Grapefruit Juice
Grilled Fish	Soup	Orange Juice
Chicken Broccoli Pasta	Applesauce	Milk
Pork Chops	Steamed Vegetables	Bottled Water
Roasted Turkey		Lemonade
Vegetable Lasagna		Iced Tea
Broiled Shrimp		
BBQ Ribs		

2. How many combo meals consisting of one main course, one side dish, and one drink are offered at *Gold Diner*? Explain how you arrived at your answer.

3. Every day you eat at *Sandwich Shop* or *Gold Diner* and order a different combo meal. How many days will it take you to order all the possible combo meals at each restaurant? Explain your reasoning.

My Notes

***Sandwich Shop* Menu**
Sandwiches
Veggie Wrap
Tuna Fish
Turkey
Chicken Breast
Side Dishes
Salad
Soup
Mixed Fruit
Drinks
Milk
Iced Tea
Apple Juice
Orange Juice
Bottled Water

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SUGGESTED LEARNING STRATEGIES: Simplify the Problem, Quickwrite, Think/Pair/Share, Vocabulary Organizer, Interactive Word Wall, Graphic Organizer

My Notes

4. *Sandwich Shop* and *Gold Diner* are going to merge into one restaurant, so a customer will be able to order a combo meal from a combined list of all the choices. How many different combo meals can be ordered at the new restaurant? Explain your reasoning.

5. Explain why the answer in Item 3 is different from the answer in Item 4.

MATH TERMS

Fundamental Counting Principle:

If there are p ways to make the first choice, q ways to make the second choice, r ways to make the third choice, and so on, then the product $p \cdot q \cdot r \cdot \ldots$ is the total number of ways a sequence of choices can be made.

The **Fundamental Counting Principle** is a useful way to count outcomes, especially in situations where it is impractical or even impossible to list them all.

6. A class has 4 students.

a. Use the boxes below to represent the seats for these 4 students. Write in each box the number of students that the teacher will choose from as she assigns each seat, beginning with Seat 1.

Seat 1	Seat 2	Seat 3	Seat 4

b. Use the seating diagram above and the Fundamental Counting Principle to determine the total number of ways that the teacher can assign the seats.

ACADEMIC VOCABULARY

A **factorial** is the product of a natural number, n, and all natural numbers less than n, written as $n!$.

$n! = n(n-1)(n-2) \cdot \ldots \cdot 2 \cdot 1.$

Zero factorial is defined as 1, or $0! = 1$.

c. Write your answer in **factorial** notation.

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SUGGESTED LEARNING STRATEGIES: Simplify the Problem, Graphic Organizer, Discussion Group

My Notes

7. A class has 20 students and 5 rows of 4 seats.

a. Write in each box the number of students that the teacher will choose from as she assigns each seat in the first row.

	Seat 1	Seat 2	Seat 3	Seat 4
Row 1				

b. In how many ways can the teacher assign 4 of the 20 students to the seats in the first row?

8. Complete the diagram below for Row 2 as you did in Item 7. Then use it and the Fundamental Counting Principle to find the number of ways that the teacher can assign 4 students to the seats in the second row, after assigning 4 students to the seats in the first row.

	Seat 1	Seat 2	Seat 3	Seat 4
Row 2				

CONNECT TO AP

In AP Statistics, counting methods such as permutations and combinations are used when solving probability problems in which the sample space is very large and it is not feasible to write the entire sample space.

9. Consider the other rows of seats in the classroom.

a. Now that the teacher has seated eight students in the first 2 rows, in how many ways can the teacher seat the next 4 students in the seats in the third row?

b. Now that the teacher has seated 12 students in the first 3 rows, in how many ways can the teacher seat the next 4 students in the seats in the fourth row?

c. Now that the teacher has seated 16 students in the first 4 rows, in how many ways can the teacher seat the next 4 students in the seats in the fifth row?

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SUGGESTED LEARNING STRATEGIES: Look for a Pattern, Vocabulary Organizer, Note-taking, Marking the Text, Interactive Word Wall, Think/Pair/Share

My Notes

10. In how many ways can the teacher seat all 20 students in the 20 seats?

Placing students in seats is an example in which order is important. One seating arrangement for the first row of seats is shown below.

	Seat 1	Seat 2	Seat 3	Seat 4
Row 1	Al	Jo	Ty	Le

A different arrangement for the first row of seats is shown below.

	Seat 1	Seat 2	Seat 3	Seat 4
Row 1	Al	Le	Ty	Jo

In mathematics, an ordered arrangement of items is called a **permutation** of the set of items. A permutation of n distinct things taken r at a time is expressed by the **permutation notation** ${}_nP_r$.

11. Use permutation notation ${}_nP_r$ to express the number of ways that the teacher can assign students to each row as described in Items 6–8. For example, the number of ways seat assignments can be made for Row 1 is expressed as ${}_{20}P_4$.

a. Row 2

b. Row 3

c. Row 4

d. Row 5

ACADEMIC VOCABULARY

permutation

WRITING MATH

You write the *permutation notation* for 30 things taken 5 at a time as ${}_{30}P_5$. A contextual example of this is finding the total number of ways to seat students from a class of 30 students in a single row of 5 seats.

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SUGGESTED LEARNING STRATEGIES: Think/Pair/Share, Look for a Pattern, Quickwrite, Simplify a Problem, Group Presentation

My Notes

The class arrangement of seats has 5 rows of 4 seats. Sometimes counting can be carried out in different ways. For example, looking at the seats in the classroom from another perspective, the classroom has 4 columns, and each column has 5 seats.

12. Use permutation notation to express the number of seating choices the teacher can make from the class of 20 students for the first column of seats.

13. Does ${}_{20}P_4 \cdot {}_{16}P_4 \cdot {}_{12}P_4 \cdot {}_{8}P_4 \cdot {}_{4}P_4 = {}_{20}P_5 \cdot {}_{15}P_5 \cdot {}_{10}P_5 \cdot {}_{5}P_5$? Explain why or why not.

14. The general formula for a permutation of n distinct things taken r at a time is

$${}_nP_r = n(n-1)(n-2) \cdot \ldots \cdot (n-r+1)$$

a. Verify that the above formula gives the correct value for ${}_{20}P_4$ and ${}_{10}P_5$.

b. Verify that ${}_nP_r = \dfrac{n!}{(n-r)!}$.

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SUGGESTED LEARNING STRATEGIES: Note-taking, Create Representations

My Notes

Permutation Formula

The number of n items chosen r at a time is

$${}_nP_r = n(n-1)(n-2)\cdot\ldots\cdot(n-r+1) = {}_nP_r = \frac{n!}{(n-r)!}$$

15. A teacher asks the class to find the number of ways that the letters in their names, all in uppercase, can be placed into different arrangements, whether or not these arrangements spell a word.

a. List all the possible arrangements for the name AMY.

b. Use the Fundamental Counting Principle to verify that the list in Part a is complete.

c. Use permutations to find the number of arrangements of the letters in the name FRANK.

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SUGGESTED LEARNING STRATEGIES: Create Representations, Quickwrite, Discussion Group

My Notes

16. PIPPI is also a student in this class.

a. List all the possible arrangements for the name PIPPI.

b. What is different about the letters in PIPPI's name as compared to the letters in FRANK's name?

c. Explain why your answer to Part b will make a difference in the total number of arrangements of the letters in PIPPI's name as compared to the letters in FRANK's name.

d. For PIPPI's name, suppose that the three P's are labeled P_1, P_2, and P_3, and the two I's are labeled I_1 and I_2. How many different arrangements are there for the P's and how many different arrangements are there for the I's?

e. For PIPPI's name, suppose that the P's are labeled P_1, P_2, and P_3 and the I's are labeled I_1 and I_2 to keep track of the P's and I's when the letters are arranged differently. How many arrangements of PIPPI's name will result?

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SUGGESTED LEARNING STRATEGIES: Quickwrite, Look for a Pattern, Vocabulary Organizer, Note-taking, Interactive Word Wall, Think/Pair/Share, Self/Peer Revision

My Notes

16. (*continued*)

f. Let N be the number of ways that the letters in PIPPI can be arranged into distinguishable arrangements. Use the results of Part d and the Fundamental Counting Principle to explain what $3! \cdot 2! \cdot N$ equals.

g. Determine the value of N in Part f. How does this value compare to the answer in Part a?

MATH TERMS

The number of **distinguishable permutations** of n items is $P = \frac{n!}{p!q!r!\ldots}$, when the n items include p copies of one item, q copies of another item, r copies of a third item, and so on.

When letters in a word, as in the name PIPPI, are rearranged, some arrangements are the same because identical letters have been interchanged or permuted. These permutations are called **indistinguishable permutations**.

Unique arrangements of items are **distinguishable permutations**.

17. Give the number of distinguishable permutations of the letters in each name. Show how you have used the general rule in the box at the left to set up and count the distinguishable permutations in each name.

PIPPI

BELLE

BABBETTE

18. How many different 10-digit numbers can be formed by rearranging the digits of the number 3,644,644,622? Show your work in the My Notes space.

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SUGGESTED LEARNING STRATEGIES: Shared Reading, Graphic Organizer, Create Representations, Quickwrite, Group Presentation

My Notes

19. A class of 20 students is electing class officers. The teacher will select a nominating committee of 4 students from the class. The committee will then determine the candidates for the election.

a. Sally, Clarence, Manuel, and Tisha were selected. Who could the teacher have selected first, second, third, and fourth? Use the boxes below to give two possible orders that the teacher could have had for selecting the nominating committee members.

1st Selection	2nd Selection	3rd Selection	4th Selection

1st Selection	2nd Selection	3rd Selection	4th Selection

b. How many arrangements could the teacher have made? Show your work to explain your reasoning

c. Explain why the order in which a teacher selects the committee members is *not* important.

d. Let N be the number of ways that the teacher can make the nominating committee selections. Explain what $4! \cdot N$ equals.

e. Determine the value of N in Part d.

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SUGGESTED LEARNING STRATEGIES: Vocabulary Organizer, Note-taking, Interactive Word Wall, Think/Pair/Share, Self/Peer Revision

My Notes

MATH TIP

Recall that a permutation is defined as ${}_nP_r = \frac{n!}{(n-r)!}$.

In mathematics, collections of items without regard to order are called **combinations**. The number of combinations of n distinct things taken r at a time is denoted by ${}_nC_r$.

In Item 19e, the value of N is a combination of 20 things taken 4 at a time and can be represented by ${}_{20}C_4$. In terms of the notation for combinations and permutations, this means that $4! \cdot {}_{20}C_4 = {}_{20}P_4$.

20. Write a formula, similar to the one for permutations, for ${}_nC_r$, the number of combinations of n things taken r at a time, in terms of n and r.

CHECK YOUR UNDERSTANDING

Write your answers on notebook paper. Show your work.

1. In how many ways can the letters in the word MATH be arranged without any of the letters being repeated?
2. Find $\frac{10!}{6! \cdot 3!}$.

A committee of 3 people is selected from a group of 5 people.

3. Use permutation notation to express the number of ways the committee can be selected.
4. Find the number of committees that can be selected.

The student council is collecting movies on DVD to send to troops overseas. Allie has 28 movies on DVD. She has decided to donate half of her movies to the student council collection.

5. Express the number of ways that Allie could choose the movies to be donated by using notation for combinations.
6. Use a calculator to find the number of combinations.
7. MATHEMATICAL REFLECTION What did you learn from doing this investigation? What questions do you still have?

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Combinations and Permutations
Pick It or Skip It

ACTIVITY 4.6

SUGGESTED LEARNING STRATEGIES: Shared Reading, Marking the Text, Activating Prior Knowledge, Group Presentation

My Notes

Games of chance are used by some states to raise money for state services to benefit people. There are many types of games of chance. Examine a few games to see what the chances of winning each game really are.

Deuce: The player must choose 2 different letters from the alphabet. To win, the player must match the first and second letters drawn in the game in the correct order.

1. How many different ways can the 2 letters be chosen?

2. If you play one time, what is the *probability* of winning at Deuce?

MATH TIP

If all outcomes in a finite sample space are equally likely to occur, then the *probability* of an event *A* is the ratio of the number of outcomes in event *A* to the total number of outcomes in the sample space. $P(A) = \frac{\text{Outcomes in event } A}{\text{Outcomes in sample space}}$

Pick-em: The player chooses 3 different numbers from 0 to 9, for example, 0-7-8. If the same 3 numbers are drawn in the game, in any order, the player wins.

3. In how many ways can the 3 numbers be selected?

4. If you play one time, what is the probability of winning Pick-em?

5. In **Straight**, the player chooses 4 numbers from 0 to 9. To win, the player must match all 4 numbers drawn in the game in the correct order. If you play one time, what is the probability of winning?

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SUGGESTED LEARNING STRATEGIES: Shared Reading

My Notes

ACADEMIC VOCABULARY

The number of **combinations** of n distinct things taken r at a time can be written as $_nC_r = \frac{n!}{r!(n-r)!}$.
An alternative notation is $\binom{n}{r} = \frac{n!}{r!(n-r)!}$.

Pick It: The player chooses 5 numbers out of 20. The order does not matter. If the player matches exactly 3 of the numbers drawn in the game, the player wins.

6. How many possible **combinations** of numbers can be drawn?

7. The numbers selected were 1, 4, 12, 16, and 19. Write 6 different possible winning tickets. What do the tickets you wrote have in common?

8. How many ways are there to match 3 numbers out of the 5 selected by the game?

9. How many numbers out of 20 were not selected in the drawing?

10. How many ways are there to match the 2 numbers not selected in the drawing out of the number you gave as your answer to Item 9?

11. What is the probability of winning this game if you play once?

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SUGGESTED LEARNING STRATEGIES: Simplify the Problem, Think/Pair/Share, Quickwrite, Group Presentation, Look For a Pattern

My Notes

12. You can win a better prize by matching exactly 4 of the numbers. Determine the probability of winning this prize.

13. How does the probability in Item 11 compare to that of Item 12?

14. You can win the best prize by matching exactly 5 of the numbers. What is the probability of winning the best prize? Explain.

15. What is the probability of winning the Pick-It game if "pick 3 or more" is considered a win?

16. Find the values of each ${}_nC_r$ and place them in a triangular pattern similar to the one given.

$$\binom{0}{0}$$

$$\binom{1}{0} \binom{1}{1}$$

$$\binom{2}{0} \binom{2}{1} \binom{2}{2}$$

$$\binom{3}{0} \binom{3}{1} \binom{3}{2} \binom{3}{3}$$

$$\binom{4}{0} \binom{4}{1} \binom{4}{2} \binom{4}{3} \binom{4}{4}$$

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SUGGESTED LEARNING STRATEGIES: Look For a Pattern, Quickwrite, Discussion Group

My Notes

The triangular pattern that you created is called Pascal's Triangle. It has many interesting patterns.

17. Write the numbers that will fill in the next row. How did you determine what the numbers would be?

18. Expand each binomial.

$(a + b)^0 =$ ________________

$(a + b)^1 =$ ________________

$(a + b)^2 =$ ________________

$(a + b)^3 =$ ________________

$(a + b)^4 =$ ________________

19. How do the coefficients of the expanded binomials relate to the numbers in Pascal's Triangle?

20. What patterns do you notice in the exponents of the expanded binomials in Item 18?

21. How does the number of terms in the expansion of $(a + b)^n$ relate to the degree n?

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SUGGESTED LEARNING STRATEGIES: Note-taking, Vocabulary Organizer, Create Representations, Think/Pair/Share, Simplify the Problem, Discussion Group

My Notes

The Binomial Theorem

For any positive n, the binomial expansion is:

$$(a+b)^n = \binom{n}{0}a^n b^0 + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \ldots \binom{n}{n}a^0 b^n.$$

22. Write the Binomial Theorem using summation notation and $\binom{n}{k}$ to represent the combination.

$$(a+b)^n = \sum_{k=0}^{n}$$

To find the r^{th} term of any binomial expansion $(a+b)^n$, use the expression $\binom{n}{r-1}a^{n-(r-1)}b^{r-1}$.

23. Find the coefficient of the 4^{th} term in the expansion of $(x-3)^8$.

24. Find the coefficient of the 6^{th} term in the expansion of $(x+2)^{11}$.

25. Find the 7^{th} term in the expansion of $(x+4)^9$.

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SUGGESTED LEARNING STRATEGIES: Create Representations, Discussion Group

My Notes

26. Use the Binomial Theorem to write the binomial expansion of $(x + 4)^7$.

27. Use the Binomial Theorem to write the binomial expansion of $(x - 4)^7$.

CHECK YOUR UNDERSTANDING

Write your answers on notebook paper. Show your work.

You selected 10 songs for a playlist on your MP3 player. The MP3 player is set to play the songs at random. The player will play all 10 songs without repeating any one song.

1. What is the probability that the songs will be played in the exact order that they are listed in the playlist?

A jar contains 40 marbles. There are 15 red and 25 yellow marbles.

2. What is the probability that if you draw 5 marbles from the jar without replacement, 3 are red?
3. What is the probability that if you draw 7 marbles from the jar without replacement, at least 5 are yellow?
4. Find the coefficient of the 5^{th} term in the expansion of $(x + 1)^6$.
5. Find the 7^{th} term in the expansion of $(x - 2)^{13}$.
6. Use the Binomial Theorem to write the binomial expansion of $(x + y)^5$.
7. **MATHEMATICAL REFLECTION** What did you learn from doing this investigation? What questions do you still have?

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Binomial Probability
Are You My Type?

ACTIVITY
4.7

SUGGESTED LEARNING STRATEGIES: Marking the Text, Activating Prior Knowledge, Close Reading

My Notes

Janet and Bob both have Type A blood. Each carries the dominant gene for the Type A antigen and the recessive gene for the Type O antigen. A Punnett Square that represents the possible gene combinations for their children is shown below.

	A	O
A	AA	AO
O	AO	OO

CONNECT TO SCIENCE

Antigens are antibody-producing proteins found on the surface of red blood cells. The type of antigen, A, B, or O, on the surface of a person's red blood cells determines that person's blood type.

A gene combination of AA or AO represents a child with Type A blood. A gene combination of OO represents a child with Type O blood. Both Bob and Janet are curious about the probabilities involving the blood types of their 8 children.

1. What is the probability that a child of Janet and Bob will be Type O?

2. What is the probability that a child of Janet and Bob will be Type A?

3. What is the sum of the two probabilities in Items 1 and 2?

The probability experiment described above is an example of a **binomial experiment**. A binomial experiment has several important characteristics:

- The situation involves a fixed number of trials.
- Each trial has only two possible outcomes. For the sake of convenience, one of these outcomes is labeled a *success*, while the other outcome is labeled a *failure*.
- The trials are **independent**, meaning that the outcome of one trial does not affect the probability of success in subsequent trials.
- The probability of success remains the same for each trial.

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SUGGESTED LEARNING STRATEGIES: Quickwrite, Shared Reading, Discussion Group

My Notes

4. Explain how finding the probability that 3 out of 8 of Bob and Janet's children will have Type O blood is a binomial experiment.

MATH TIP

The probability of successive independent events *A*, *B*, *C*, ... occurring is $P(A \text{ and } B \text{ and } C \text{ and} \ldots) = P(A) \cdot P(B) \cdot P(C) \cdot \ldots$

To determine the probabilities in a binomial experiment like the one described above, it is helpful to consider a simple probability experiment consisting of tossing a fair coin 3 times.

5. Explain how tossing a fair coin 3 times is a binomial experiment.

6. Find the probability of obtaining a head, a head, and then a tail, in that order, when tossing a coin 3 times in the following two ways.

a. List all of the outcomes in the sample space.

b. Apply the probability rules for successive independent events.

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SUGGESTED LEARNING STRATEGIES: Discussion Group, Shared Reading, Quickwrite

My Notes

7. Consider the question, "What is the probability of obtaining exactly 2 heads from 3 coin tosses?" How does this question differ from the situation in Item 6?

8. Find the probability of obtaining exactly 2 heads from 3 tosses of a coin in the following two ways.

a. List all of the outcomes in the sample space.

b. Apply the appropriate probability rules.

MATH TIP

The probability of *A* or *B* occurring if *A* and *B* are mutually exclusive events is $P(A \text{ or } B) = P(A) + P(B)$.

The probability of *A* or *B* occurring if *A* and *B* are not mutually exclusive events is $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

A fast food restaurant wants to increase customer interest in a new chicken sandwich. They are offering one scratch-off card with each purchase of a chicken sandwich. Each scratch-off card has a 20% chance of being a winning card, and a customer has collected three cards from previous purchases.

9. Explain why this situation represents a binomial experiment.

10. What is the probability of having the first scratch-off card be a winner, the second card be a winner, and the third card be a loser? Explain your reasoning.

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SUGGESTED LEARNING STRATEGIES: Quickwrite, Activating Prior Knowledge, Discussion Group, Group Presentation

My Notes

11. Consider the question, "What is the probability that a customer has exactly 2 winning scratch-off cards out of 3 cards?" How does this question differ from the situation in Item 10?

12. What is the probability that a customer has exactly 2 winning scratch-off cards out of 3 cards?

13. In probability experiments, sometimes the order in which the events occur is important. At other times it is not. Which two of the previous situations are examples of a probability experiment where the order in which successive events occur is *not* important?

14. What counting method can you use to determine the total number of possibilities in probability experiments where the order is not important?

In the fast food situation, suppose that a customer has collected 5 scratch-off cards and wants to know the probability that exactly 2 cards will be winners.

15. How many different ways can a customer have 2 winning cards out of the 5 scratch-off cards? Use the counting method you identified in Item 14.

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SUGGESTED LEARNING STRATEGIES: Discussion Group, Group Presentation

My Notes

16. List all of the different outcomes for having 2 winning cards out of the 5 scratch-off cards. One outcome is shown below. Does the number of outcomes in your list agree with your answer to Item 15?

WWLLL

17. Find the probability of each of the possible outcomes listed in Item 16.

18. Use your answers to Items 15 and 17 to find the probability of a customer having exactly 2 winning cards out of 5 scratch-off cards.

19. In Item 18, you found the probability of a customer having 2 winning cards out of 5 scratch-off cards. What are all the possible numbers of winning cards that a customer can have with 5 scratch-off cards?

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SUGGESTED LEARNING STRATEGIES: Create Representations, Vocabulary Organizer

My Notes

MATH TERMS

A *discrete random* variable may take on only a countable number of distinct values such as 0, 1, 2, 3, 4, ...

ACADEMIC VOCABULARY

A **probability distribution** describes the *values* and *probabilities* associated with a random event. The values must cover all of the possible outcomes of the event and the sum of all the probabilities must be exactly one.

20. In this situation, the number of winning scratch-off cards is called a **discrete random variable**, X. A **probability distribution of X** lists the possible number of successes X and their associated probabilities, $P(X)$.

a. Find the probability of a customer having exactly 3 winning cards out of a total of 5 scratch-off cards. Show your method.

b. Complete the table below to create the probability distribution of x winning cards out of 5 scratch-off cards.

X (number of winning cards)	**0**	**1**	**2**	**3**	**4**	**5**
$P(X)$ (the probability of exactly X winning cards out of 5 scratch-off cards)						

21. Consider a binomial experiment with a probability of success equal to p. The notation $P(k)$ represents the probability of k successes in n trials. Write an expression below that gives the value of $P(k)$, for n trials.

$P(k) =$

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SUGGESTED LEARNING STRATEGIES: Identify a Subtask, Quickwrite

My Notes

22. Remember that Bob and Janet have 8 children. Determine the following probabilities. Write your answers in the My Notes space. Show your work.

a. They have 3 children with Type O blood.

b. They have 5 children with Type O blood.

c. They have 7 children with Type O blood.

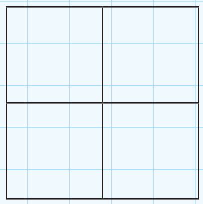
23. Recall that the probability of a child of Bob and Janet having Type O blood is 25%.

a. Which probability from Item 22 was the largest?

b. Find 25% of 8 and use this value to explain why your answer in Part a is reasonable.

c. Without calculating the probabilities, which of the following would have the largest probability among 8 children: 1 with Type O blood, 2 with Type O blood, or 3 with Type O blood. Explain your reasoning.

24. Elisha and Ismael have 6 children. Elisha carries the genes for Type A antigens and Type B antigens. Ismael carries the genes for Type O antigens only. Given that the genes for Type O antigens are recessive to genes for Type A and Type B antigens, what is the probability that 3 of their children have Type B blood? Use the Punnett Square at the right to help answer this question.



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CHECK YOUR UNDERSTANDING

Write your answers on a separate sheet of notebook paper. Show your work.

1. Explain how tossing a fair coin 6 times is a binomial experiment.
2. Find the probability of getting 4 heads in 6 tosses of a fair coin.
3. Find the probability of getting 4 or more heads in 6 tosses of a fair coin.

An archer shoots 8 arrows at a target. Assume that each of her shots are independent and that each have the probability of hitting the bull's-eye of 0.7.

4. What is the probability that she hits the bull's-eye exactly 4 times?
5. What is the probability that she hits the bull's-eye at least 4 times?
6. **MATHEMATICAL REFLECTION** What did you learn from doing this investigation? What questions do you still have?

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Combinations, Permutations, and Probability

THE WEDDING

Embedded Assessment 3

Use after Activity 4.7.

Brad and Janet are getting married. They have a wedding party of 5, with 3 bridesmaids and 2 groomsmen.

1. In how many different ways can 4 of the 5 members of the wedding party line up for a photo?

2. If 3 from the group of 5 wedding party members are chosen at random for another picture, in how many ways can this be done?

Brad and Janet have family members plus 25 guests coming to the wedding. They plan on seating the family in the front row, but they will seat the rest of the guests randomly.

3. What is the probability that Janet's 2 best friends will be selected to sit in the first two available seats in the second row?

4. Brad has 7 close friends. What is the probability that 2 of his close friends sit in the first 2 available seats in the second row?

5. Use the Binomial Theorem to expand $(x + 3)^5$.

6. There are 2 types of wedding favors, a white candle and a black candle, being given to the guests. Each guest is equally likely to get either candle. If 10 people are given wedding favors, what is the probability that 7 people will receive black candles?

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Embedded Assessment 3

Use after Activity 4.7.

Combinations, Permutations, and Probability

THE WEDDING

	Exemplary	Proficient	Emerging
Math Knowledge #5	The student correctly uses the Binomial Theorem. (5)	The student finds the correct product, but does not use the Binomial Theorem. OR The expansion is partially correct, using the Binomial Theorem.	The student gives an incorrect expansion.
Problem Solving #1, 2, 3, 4, 6	The student: • Gives the correct number of different ways. (1, 2) • States the correct probability. (3, 4, 6)	The student: • Gives the correct number of different ways for one of question 1 or 2, but not both. • States the correct probability for two of the three questions.	The student: • Gives the correct number of different ways for neither question. • States the correct probability at least one of the three questions.

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ACTIVITY 4.1

Decide if each function is a polynomial. If it is, write the function in standard form. Then state the degree and leading coefficient.

1. $f(x) = 7x^2 - 9x^3 + 3x^7 - 2$
2. $f(x) = 2x^3 + x - 5^x + 9$
3. $f(x) = x^4 + x + 5 - \frac{1}{4}x^3$
4. $f(x) = -0.32x^3 + 0.08x^4 + 5x^{-1} - 3$

Describe the end behavior of each function.

5. $f(x) = -4x^4 + 5x^3 + 2x^2 - 6$
6. $f(x) = x^{13} + 7x^{12} - 13x^5 + 12x^2 - 6$
7. A cylindrical can is being designed for a new product. The height of the can plus twice its radius must be 45 cm.
   a. Find an equation that represents the volume of the can, given the radius.
   b. Find the radius that yields the maximum volume.
   c. Find the maximum volume of the can.

ACTIVITY 4.2

Find each sum or difference.

8. $(4x^3 + 14) + (5x^2 + x)$
9. $(2x^2 - x + 1) - (x^2 + 5x + 9)$
10. $(5x^2 - x + 10) + (12x - 1)$
11. $(7x^2 - 11x + 5) - (12x^2 + 8x + 19)$

Find each product.

12. $5x^2(4x^2 + 3x - 9)$
13. $(x + 2)(3x^3 - 8x^2 + 2x - 7)$

Find each quotient, using long division.

14. $\frac{x^4}{(x + 1)^3}$
15. $(2x^3 - 3x^2 + 4x - 7) \div (x - 2)$

Find each quotient, using synthetic division.

16. $(2x^3 - 4x^2 - 15x + 4) \div (x + 3)$
17. $\frac{x^3 - x^2 - 14x + 11}{x - 2}$

ACTIVITY 4.3

18. Factor by grouping.
   a. $25x^3 + 5x^2 + 30x + 6$
   b. $28x^3 + 16x^2 - 21x - 12$
19. Use the pattern of a difference or a sum of cubes to factor each expression.
   a. $125x^9 + y^3$
   b. $x^3 - 216y^6$
20. Factor, using quadratic patterns.
   a. $x^4 - 7x^2 + 6$
   b. $x^4 - 4x^2 + 3$
   c. $x^6 - 100$
21. Find the zeros of each function by factoring and using the Zero Product Property.
   a. $f(x) = x^3 - 1331$
   b. $g(x) = -4x^3 + 20x^2 + 56x$
   c. $h(x) = 3x^3 - 36x^2 + 108x$
22. Write a polynomial function of n^{th} degree, given real or complex roots.
   a. $n = 4; x = -3, x = 2i, x = 4$
   b. $n = 3; x = -2, x = 1 + 2i$

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Practice

ACTIVITY 4.4

Determine the end behavior of each function.

23. $y = 4x^7 - 2x^3 + 8x + 6$

24. $y = -3x^{11} + 4x^9 - x^4 + 10x^3 + 9$

Use what you know about end behavior and zeros to graph each function.

25. $y = x^4 + 2x^3 - 43x^2 - 44x + 84$
$= (x - 1)(x - 6)(x + 2)(x + 7)$

26. $y = x^5 - 14x^4 + 37x^3 + 260x^2 - 1552x - 2240$
$= (x - 7)(x + 5)(x - 4)^3$

27. Determine all the possible rational zeros of $f(x) = -4x^3 - 13x^2 - 6x - 3$.

28. Graph $f(x) = -4x^3 - 13x^2 - 6x - 3$.

29. Determine the possible number of positive and negative real zeros for $h(x) = 2x^3 + x^2 - 5x + 2$.

30. Graph $h(x) = 2x^3 + x^2 - 5x + 2$.

31. Solve the inequality $-x^4 + 20x^2 - 32 \geq 32$.

ACTIVITY 4.5

32. In how many ways can the numbers 1, 2, 3, 4, and 5 be arranged without any of the numbers being repeated?

33. Find $\frac{8!}{4! \cdot 3!}$.

A basketball team of 5 players is being selected from a group of 20 players.

34. Use permutation notation to express the number of ways that the team can be selected.

35. Find the number of team configurations that can be selected.

36. Give the number of distinguishable permutations of the name JEANNETTE.

37. A pizzeria offers a vegetarian pizza with a choice of any three different vegetable toppings from a list of eight. How many different vegetarian pizzas can be ordered? Show your work.

ACTIVITY 4.6

Cards are drawn at random from a standard deck of 52 cards, without replacement.

38. If two cards are drawn, what is the probability that both cards are jacks?

39. If two cards are drawn, what is the probability that both cards are hearts?

40. If four cards are drawn, what is the probability that two cards are hearts?

Use this information for Items 41–42. A jar contains 50 marbles. Twenty are red, 10 are yellow and 20 are green.

41. What is the probability that if you draw 8 marbles from the jar without replacement, 5 are green?

42. What is the probability that if you draw 8 marbles from the jar without replacement, 6 are yellow?

43. Find the coefficient of the 3rd term in the expansion of $(x^2 + 2)^5$.

44. Find the 6th term in the expansion of $(4x - 3)^{10}$.

45. Use the Binomial Theorem to write the binomial expansion of $(3x - y)^5$.

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ACTIVITY 4-7

Seventy-five percent of a population are children. A sample of 16 people is selected with replacement from the population.

46. Explain how this situation represents a binomial experiment.

47. Find the probability that 5 people selected from the sample are adults.

48. Find the probability that 7 people selected from the sample are adults.

49. What is the most likely number of adults in the sample?

50. What is the probability of getting exactly the number you found in Item 49?

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Reflection

An important aspect of growing as a learner is to take the time to reflect on your learning. It is important to think about where you started, what you have accomplished, what helped you learn, and how you will apply your new knowledge in the future. Use notebook paper to record your thinking on the following topics and to identify evidence of your learning.

Essential Questions

1. Review the mathematical concepts and your work in this unit before you write thoughtful responses to the questions below. Support your responses with specific examples from concepts and activities in the unit.
   - How do polynomial functions help to model real-world behavior?
   - How is probability used in real-world settings?

Academic Vocabulary

2. Look at the following academic vocabulary words:
   - combination
   - end behavior
   - extrema
   - factorial
   - permutation
   - polynomial function
   - probability distribution

   Choose three words and explain your understanding of each word and why each is important in your study of math.

Self-Evaluation

3. Look through the activities and Embedded Assessments in this unit. Use a table similar to the one below to list three major concepts in this unit and to rate your understanding of each.

Unit Concepts	Is Your Understanding Strong (S) or Weak (W)?
Concept 1	
Concept 2	
Concept 3	

   a. What will you do to address each weakness?
   b. What strategies or class activities were particularly helpful in learning the concepts you identified as strengths? Give examples to explain.

4. How do the concepts you learned in this unit relate to other math concepts and to the use of mathematics in the real world?

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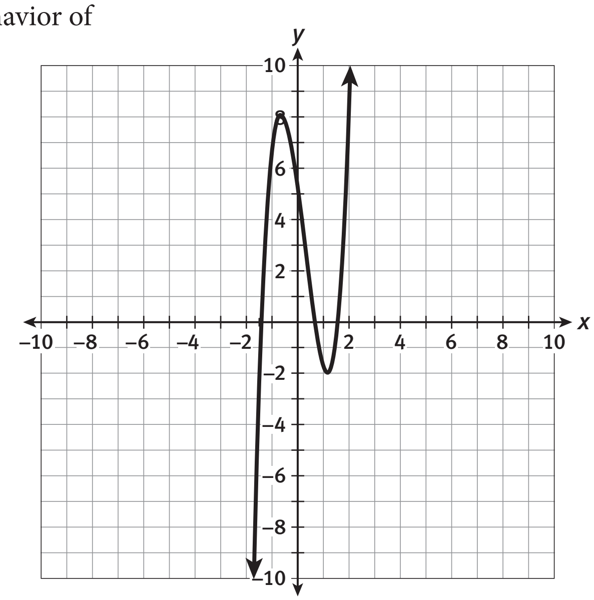
1. Given the graph of $f(x) = 4x^3 - 3x^2 - 8x + 5$, which statement about the end behavior of $f(x) = 4x^3 - 3x^2 - 8x + 5$ is true?

A. $f(x) \to +\infty$ as $x \to 0$

B. $f(x) \to +\infty$ as $x \to -\infty$

C. $f(x) \to -\infty$ as $x \to +\infty$

D. $f(x) \to -\infty$ as $x \to -\infty$

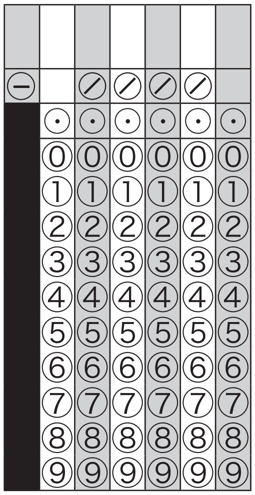


1. Ⓐ Ⓑ Ⓒ Ⓓ

2. What is the remainder for the following?

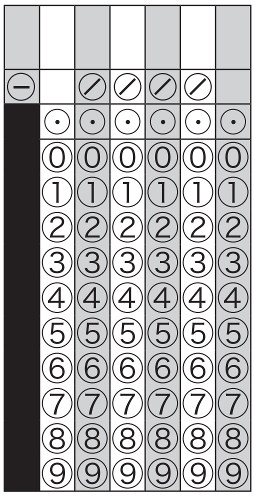
$(5x^3 - 2x^2 + 7) \div (x - 3)$

2.



3. Given the function $f(x) = x^3 - 2x^2 - x + 2$, what is the **sum** of the zeros of the function?

3.



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Read
Solve
Explain

4. An object is projected vertically upward from ground level with a velocity of 352 feet per second. The height ***h*** after ***t*** seconds is given by the function below:

$$h(t) = -16t^2 + 352t$$

a. Find when the object reaches the maximum height and determine that height. Show work to support your answer.

Answer and Explain

b. Give the interval(s) of time over which the height is increasing and the interval(s) of time over which it is decreasing.

Answer and Explain

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